The Research-Driven Solution to Raise the Quality of High School Core Courses

QualityCore®

The Purpose and Predictability of Patterns

Algebra II

Model Instructional Unit 1

ACT®

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Note

QualityCore® Instructional Units illustrate how the rigorous, empirically researched course standards can be incorporated into the classroom. You may use this Instructional Unit as is, as a model to assess the quality of the units in use at your school, or as a source of ideas to develop new units. For more information about how the Instructional Units fit into the QualityCore program, please see the *Educator's Guide* included with the other QualityCore materials.

ACT recognizes that, as you determine how best to serve your students, you will take into consideration your teaching style as well as the academic needs of your students; the standards and policies set by your state, district, and school; and the curricular materials and resources that are available to you.

Unit 1 The Purpose and Predictability of Patterns

Purpose

The purpose of this unit is to connect students' prior knowledge of numeric and geometric patterns to the study of sequences and series. Students will move from general pattern-seeking, through explorations of the properties of arithmetic and geometric sequences, to analyses of sequences and series and applications of them to real-world problems. Along the way, they will learn notation and terminology associated with patterns, particularly the sigma notation for series.

Overview

Pattern-seeking and inductive reasoning are vital skills, especially when they are used to make predictions. The study of sequences helps students begin to conceptualize natural processes in terms of functions and equations. The National Council of Teachers of Mathematics (NCTM) (2000, pp. 158, 296) has emphasized the importance of studying numerical and geometric patterns across grade levels, arguing that students should be able to

- describe and extend geometric and numeric patterns;
- represent and analyze patterns and functions, verbally, symbolically, and visually; and
- generalize patterns using explicitly and recursively defined functions.

In this introductory unit of Algebra II, students will accomplish these goals by exploring real-world applications of patterns, sequences, and series. They will learn how appropriate mathematical language facilitates modeling and solving problems involving sequences and series, which provide a natural introduction to functions whose domain is the natural numbers. In addition, by exploring the effect of varying a numeric or geometric pattern, students will gain deep understanding of the role of sequences and series in everyday life and develop the ability to make predictions based upon observations of the patterns they see.

By having students work both independently and cooperatively to explore the connections between the world around them and the mathematical concepts of sequences and series, the introductory unit in Algebra II is designed to help teachers establish and maintain rigorous expectations in the classroom. However, if students are to meet those expectations, they must do most of the explaining and demonstrating. In such a student-centered classroom, a teacher's role is to develop and expand students' thinking by posing rich problems and investigations, listening, observing, questioning, and understanding the students—actions that, when emphasized, create a classroom atmosphere of collaboration and achievement.

Time Frame

This unit requires approximately eleven 45–50 minute class periods.

We are usually convinced more easily by reasons we have found ourselves than by those which have occurred to others.
—Blaise Pascal, *Pensées* (1670/1995, p. 228)

In the usage of Algebra, the pattern of the marks on paper is a particular instance of the pattern to be conveyed to thought.
—Alfred North Whitehead (1948, p. 82)

The mathematician's patterns, like the painter's or poet's, must be beautiful; *the ideas, like the colours or the words, must fit together in a harmonious way.*
—G. H. Hardy, *A Mathematician's Apology* (1940/1992, p. 85)

UNIT 1
THE PURPOSE AND PREDICTABILITY OF PATTERNS

Prerequisites

- Identify and use properties of real numbers and the correct order of operations to simplify expressions
- Use inductive reasoning to make conjectures and deductive reasoning to arrive at valid conclusions

Selected ACT Course Standards

The primary standards, which represent the central focus of this unit, are listed below and highlight skills useful not only in Algebra II, but in other disciplines as well. Secondary standards are listed in Appendix L.

B.1. Mathematical Processes

d. Use the language of mathematics to communicate increasingly complex ideas orally and in writing, using symbols and notations correctly
e. Make appropriate use of estimation and mental mathematics in computations and to determine the reasonableness of solutions to increasingly complex problems
f. Make mathematical connections among concepts, across disciplines, and in everyday experiences
g. Demonstrate the appropriate role of technology (e.g., calculators, software programs) in mathematics (e.g., organize data, develop concepts, explore relationships, decrease time spent on computations after a skill has been established)

H.2. Sequences and Series

a. Find the *n*th term of an arithmetic or geometric sequence
b. Find the position of a given term of an arithmetic or geometric sequence
c. Find sums of a finite arithmetic or geometric series

d. Use sequences and series to solve real-world problems
e. Use sigma notation to express sums

Research-Based Strategies

- Cues and Questions (pp. 6, 16)
- Quick Write (p. 9)
- 3-2-1 Assessment (p. 10)
- Think-Pair-Share (pp. 11, 16, 19, 23)
- Reflective Questioning (pp. 13, 17)
- Hand Signals (p. 13)
- Graphic Organizer (pp. 20, 29)
- Index Cards (p. 27)
- WebQuest (p. 32)

Essential Questions

1. When, how, and why are patterns useful in our everyday lives?
2. How do you know when an observed pattern forms a sequence?
3. How are patterns useful for making predictions?

Suggestions for Assessment

Except where otherwise noted, assessments can be given a point value or they can simply be marked off as completed.

Preassessments

Worksheet—The Prime Factors worksheet (p. B-3) assesses students' facility finding the prime factorization of numbers. (Day 1)

Homework—The Number Patterns homework (p. B-6) assesses students' prior knowledge of geometric and numeric patterns. (Day 1)

Homework—The Algebra Skills homework (pp. C-12–C-13) assesses students' prior knowledge of integer arithmetic, linear equations and inequalities, and basic coordinate geometry. (Day 2)

Embedded Assessments

Activity—The Station Problems activity (pp. C-2–C-6) allows students to explore the connections between patterns and the world around them. (Day 2)

Rubric—The Group Participation and Collaboration Rubric (p. C-11) provides a way to assess students' small-group work. In addition, it can be used to emphasize the fact that working well with others is a useful skill in both school and later life.

Worksheet—Use the Simon Says worksheet (p. D-2) to introduce arithmetic and geometric sequences in the real world. (Day 3)

Homework—Simon Says 2 (pp. D-10–D-11) uses a familiar problem to give students practice identifying an arithmetic sequence and the associated vocabulary. (Day 3)

Worksheet—The Movie Time worksheet (p. E-4) introduces students to finding a term of an arithmetic sequence using a real-world example. (Day 4)

Homework—Even and Odd (pp. E-9–E-10) gives students practice using the formulas for sequences and series. (Day 4)

Homework—The Arithmetic Sequences and Series Practice homework (p. F-9) assesses students' understanding of arithmetic sequences and series. (Day 5)

Worksheet—The Growing Geometrically worksheet (p. G-4) and the Shrinking Geometrically worksheet (p. G-6) introduce geometric sequences by exploring real-world problems. (Day 6)

Worksheet—The Sigma Notation worksheet (p. G-8) introduces sigma notation through examples of both geometric and arithmetic series. (Day 6)

Worksheet—The Summing Geometrically worksheet (p. H-4) introduces methods of summing geometric series. (Day 7)

Homework—The Geometric Sequences and Series Practice homework (p. I-6) gives students practice using the formulas for geometric sequences and series. (Day 8)

Worksheet—The Sequences, Series, and Salaries worksheet (pp. J-2–J-3) compares arithmetic and geometric growth in salaries. (Days 9–11)

Worksheet—The Sequences, Series, and Patterns worksheet (pp. J-5–J-6) connects geometric sequences and series to fractal geometry—a hot topic in contemporary mathematics—through the construction of a snowflake curve. (Days 9–11)

Worksheet—The Sequences and Series Review worksheet (pp. J-9–J-10) provides an opportunity for students to review the concepts presented in the unit. (Days 9–11)

Unit Assessment

Test—The Sequences and Series Test (pp. J-12–J-13) assesses the concepts discussed in class. (Day 11)

Unit Description

Introduction

Materials & Resources

- ❑ Unit Assignments and Assessments (pp. A-2–A-3)

On the first day of Algebra II, students are likely to have well-established attitudes about mathematics. While some students will already have fallen in love with the subject, others will be convinced that they cannot do mathematics at all. Because students normally come to Algebra II having just completed a course in Geometry, they often betray a decided preference for one subject over the other. Their performance often reflects this. Regardless of students' attitudes toward mathematics, the first days of Algebra II afford all students the opportunity to make a fresh start. Students can build confidence in their ability and skill in working with algebraic concepts through enjoyable, engaging activities that demonstrate the relevance of mathematics to their everyday lives. The exploration of numeric and geometric patterns affords an excellent launchpad for Algebra II for three reasons:

1. Patterns have rich connections to real-world phenomena.
2. Patterns offer students the same intriguing, engaging challenges that lie at the heart of popular puzzles such as Rubik's Cube® and Sudoku.
3. Patterns help build confidence and skill in inductive reasoning and algebraic concepts and techniques.

As you begin studying patterns, keep in mind that the first days of class should also focus on building classroom community. Group work, for example, helps students get to know each other by working cooperatively.

Tips for Teachers

The website of the North Central Regional Education Laboratory features useful information about collaborative learning and small groups. To support successful group work, they suggest that teachers:

- Include students of mixed abilities when forming heterogeneous groups.
- Provide specific work areas for each group.
- Clearly communicate rules and assignment requirements.
- Encourage all group members to participate.
- Acknowledge and support positive and effective work within and between groups.

Tips for Teachers

Prior to the first day of school, use the following checklist (Wright, 1989) to identify tasks not yet accomplished or to spark new ways of starting off the new year.

- ❑ Am I energized to be enthusiastic about this class?
- ❑ Is the classroom arranged properly for the day's activities?
- ❑ Are my name, course title, and room number on the board?
- ❑ Do I have an icebreaker planned?
- ❑ Do I have a way to start learning names?
- ❑ Do I have a way to gather information on student backgrounds, interests, course expectations, questions, and concerns?
- ❑ Is the syllabus complete and clear?
- ❑ Have I outlined how students will be evaluated?
- ❑ Do I have announcements of needed information for the day?
- ❑ Do I have a way of gathering student feedback?
- ❑ When the class is over, will students want to come back? Will I want to come back?
- ❑ Do I have the essential questions and unit standards on display in the classroom?

Working in groups promotes face-to-face interaction, positive interdependence, and individual accountability. It also provides opportunities for students to hear different perspectives and ideas, to share responsibilities for complex projects, to identify and appreciate the talents of others, to think through ideas in order to share them with others, and to take a vested interest in the learning of their peers—skills that are useful throughout school and later life.

To build this type of classroom community requires time, thought, and care at the beginning of the school year. According to Harry and Rosemary Wong (2004), the message that is sent to students on the first day can make or break an entire course. Often, the things that teachers do on the first day, such as listing rules and giving the entire semester's assignments, turn students off to the teacher, the course, and the discipline as a whole. In fact, there will be plenty of time in later days to address those points; the first day should instead be one in which students are introduced to you, to each other, and to the fascinating possibilities of algebra.

Therefore, preparing for students before they enter the classroom is critical—not just for a successful first week, but for a successful school year. Before class begins, identify the procedures that you expect students to follow in your classroom and be ready to model them. To build classroom rapport and to communicate that students' ideas matter, make a point to acknowledge and talk to each student. Following the example (p. A-2), fill out the Unit Assignments and Assessments record keeping log (p. A-3) to give students a clear picture of this unit's assignments. Begin the first and every class with a warm-up, either written on the board or placed at students' desks for them to work on upon entering the classroom. By piquing students' interest, focusing their attention, connecting to previous learning, or introducing the subject of the day's lesson, warm-up activities make the most of the time you have and prepare students for the day's learning. They also allow you to take attendance without wasting valuable educational time.

Activities conducted at the end of a lesson designed to wrap up students' learning are also important. They allow you and the students to assess the level of comprehension of the concepts developed in class that day. Wrap-ups can include Exit Slips, sample problems, 3-2-1 Assessments, Whiteboarding, and Journaling. Begin using warm-ups and wrap-ups on the first day of school and establish them as a routine for the entire school year.

Suggested Teaching Strategies/Procedures

Day 1

Students are introduced to the classroom, complete a mathematical autobiography, and participate in icebreakers that review prior knowledge of the prime number sequence.

Materials & Resources

- ❑ Course syllabus*
- ❑ Mathematics Autobiography (p. B-2)
- ❑ Prime number list*
- ❑ Opaque sheet or large poster board*
- ❑ Index cards*
- ❑ Prime Factors (p. B-3)
- ❑ Prime Factors Key (p. B-4)
- ❑ Introduction to Patterns in Numbers (p. B-5)
- ❑ Introduction to Number Patterns (p. B-6)
- ❑ Introduction to Number Patterns Key (p. B-7)
- ❑ Calculators*

* Materials or resources not included in published unit

Before class, place a copy of the Unit Assignments and Assessments record keeping log, the course syllabus, and the Mathematics Autobiography worksheet (p. B-2) on each desk or in a designated location in the room. In order to facilitate seating on the first day, assign each desk a unique prime number. In a class of 30 students, for example, use each of the first 30 prime numbers (2–113). After writing each number on two index cards, keep one set of the cards to pass out to students and attach the other set to the students' desks. In addition, post a list of the primes in a prominent place in the classroom. If you put the list on a large poster, then cover it with a brightly colored, opaque sheet to attract students' attention when they enter the classroom. Take mental notes of students' reaction to the covered list: Do they ask questions about it or about what it covers? Do they try to peek? Do they glance at it frequently? Use students' initial curiosity as you introduce the Prime Number activity during the day's lesson.

By treating students with respect and courtesy, you will begin to build personal relationships from the very first meeting. Therefore, stand outside your door to greet students as they enter the classroom for the first time. Welcome them to Algebra II, introduce yourself, and hand each of them one of the index cards. Ask them to sit down at the desk with the corresponding prime number, read the syllabus, and begin work on the Mathematics Autobiography worksheet. Let the students know that while you will read each of their autobiographies, the information in them will remain private. In addition, ask students to record their prime numbers somewhere safe; they will be needed for future reference.

When students have finished writing their autobiographies, collect the worksheets. Before the next class, read them, take note of useful information,

Tips for Teachers

Neal Sloane's *Online Encyclopedia of Integer Sequences* (2007) is an excellent resource for number sequences of all kinds. In order to locate a specific number sequence, simply type the first few terms into the search box. Searching for the first few prime numbers will bring up the entry on prime numbers; clicking on the word *list* near the top of the page leads to a table of the first 58 primes.

and organize them in a notebook. As you get to know the students and their mathematical abilities and interests, use what you learn to help them grow as learners and mathematicians. For example, suppose in her autobiography Chantelle listed several strategies that she uses when she is stuck on a math problem. When a difficult problem arises in class, ask Chantelle to describe her solution strategies; it might help other students to hear how a classmate approaches a mathematical challenge.

Next, initiate an icebreaker. Often, by the time students reach Algebra II, most of them are already well acquainted, and you are the only one challenged with many new names and faces. An activity adapted from Carol Apacki's book *Energize!* (1991, p. 23) works well for a group of students who already know each other well. It can be a success even in small schools where the same students have been together since kindergarten, and it is flexible enough that it can be adapted and used to learn something about students who are unacquainted with their classmates.

Before calling attention to the list of prime numbers posted in the classroom, address students' reactions to the covered list. Describe their reactions and interpret what they might mean: "When you came to class today, some of you were interested in what I was hiding. Your questions and actions demonstrate to me that you are curious and persistent. Curiosity and persistence are habits of mind that are important for mathematicians to have." Then, ask students to volunteer some of the characteristics of the primes. Students should recall that a prime number is a positive integer greater than 1 having no positive divisors other than 1 and itself. Any integer greater than 1 that is not prime is called *composite*. Students should also recall that every composite number has a unique prime factorization (the Fundamental Theorem of Arithmetic).

If students are slow to recall these facts, the practice of Cues and Questions is a highly effective research-based strategy for triggering students' memories and connecting new information to prior knowledge (Marzano, Norford, Paynter, Pickering, & Gaddy, 2001, pp. 267–269). Offer cues and ask questions that guide them toward generalizations about the factorization properties of the natural numbers. For example, following the definition of the prime numbers with a look at the numbers 2 and 3 will cue students that these numbers are primes. Additional questioning provides a sense of contrast between prime and composite numbers ("Compare the factors of 3 to the factors of 4. Do you see any differences?") and helps trigger prior knowledge of unique prime factorization ("Can you write 12 as the product of prime factors? How many ways can you do this?"). Inviting students to develop generalizations about what they already know is an important step in forming a classroom pedagogy that relies on inductive reasoning. Familiarizing students with the process of generalizing now will help later, when they will be asked to form and test hypotheses based upon their generalizations.

When students seem comfortable in the recall of basic facts about prime numbers, explain that each of us is, in some way, like a prime factor: we have unique characteristics to contribute to the class. In this activity, each student will share one of these unique characteristics. At the top of the index cards on which their seat numbers were written, each student should write his or her name and the title "Unique Prime Factor." Encourage students to think of unique personal characteristics that their peers may not know about them, to write one of those characteristics on the card, and to prepare to share this characteristic with the class. Students might include almost anything:

- Facts ("I was born in Sudan, and my parents and I are refugees.")
- Jobs ("I waitress nights to save money for college.")
- Favorite foods ("I love spicy Indian dishes.")
- Experiences ("I worked on a Habitat for Humanity project over the summer.")
- Awards ("I won a blue ribbon for photography at the state fair.")
- Skills ("I can easily spell any word backwards.")
- Interests ("I love bicycle racing.")

Then, in order from least prime number to greatest, each student should announce his or her name, prime number, and unique prime factor to the class. As they talk, take attendance and complete a seating chart. Then, collect students' index cards and add your own. Throughout the course, when a day's activities are complete and a few moments remain, randomly choose a card and read the "Unique Prime Factor" clue. Ask students to remember or guess whom the characteristic represents. Once the person is correctly identified, allow him or her to elaborate.

Now and throughout the course, take time to inform students that the prime numbers are among the most important number sequences, having wide applications in mathematics, science, and technology. Some scientists believe that the primes are so fundamental to civilization that they can be used as a means of communication with extraterrestrial life. This is the subject of a fascinating short article by the mathematician Carl Pomerance (2004). It is also the premise of Robert Zemeckis's popular film *Contact* (1997), and the Carl Sagan novel (1985) upon which the film is based. The article, novel, and film may be used as supplementary material for this unit.

The Prime Factors worksheet (p. B-3) will serve as a wrap-up for the day. Distribute a copy to each student and ask the class to spend the last few minutes working quietly on the problems. Before students leave, collect the worksheets and use them as a formative assessment of students' number skills.

Finally, distribute the Introduction to Patterns in Numbers reading (p. B-5) and the Number Patterns homework (p. B-6). Use the reading as a reference point for future discussion and reflection, and use the homework to assess students' prior knowledge of geometric and numeric patterns.

Tips for Teachers

Despite the importance of the prime number sequence, its pattern of growth is not easy to characterize: it is neither arithmetic nor geometric. Indeed, the growth and distribution of the prime numbers has been a subject of mathematical fascination since the time of Euclid (circa 300 B.C.), and lies at the heart of some of the most important open problems in mathematics and computing. Mathematicians believe that the solution to the Riemann Hypothesis—the most famous unsolved problem in all of mathematics—is the key to obtaining a precise characterization of the distribution of the prime numbers (Weisstein, 1999).

Day 2

In preparation for the study of sequences, students explore the prime number sequence, along with a variety of other numeric and geometric patterns.

Materials & Resources

- ❑ Station Problems (pp. C-2–C-6)
- ❑ Station Problems Key (pp. C-7–C-8)
- ❑ Eratosthenes' Prime Sieve (p. C-9)
- ❑ Eratosthenes' Prime Sieve Key (p. C-10)
- ❑ Calculators*
- ❑ Prime number list*
- ❑ Sticky notes*
- ❑ Poster board*
- ❑ Group Participation and Collaboration Rubric (p. C-11)
- ❑ Algebra Skills (pp. C-12–C-13)
- ❑ Algebra Skills Key (pp. C-14)

* Materials or resources not included in published unit

Before class begins, identify 5 locations in the classroom that will serve as stations for the Station Problems activity (pp. C-2–C-6). Number the stations from 1 to 5, and place sufficient copies of the corresponding worksheet at each station so that every student in the class will have a copy.

Greet students as they enter the classroom and instruct them to form groups of four by arranging themselves in ascending prime order, according to the prime numbers they were assigned on Day 1. Thus, one group will consist of the students with numbers 2, 3, 5, and 7; another of 11, 13, 17, 19; and so forth. (If the number of students in the class is not evenly divisible by 4, allow a few groups of either 3 or 5 students each.) The students will be working in these prime clusters for the rest of the day, which will allow them to check, discuss, and validate each other's thinking. Direct students' attention to the overhead projector or the board. Ask them to warm up by working in their groups on the Eratosthenes' Prime Sieve activity (p. C-9), using calculators if they are available. While students work, circulate through the classroom, taking attendance and collecting and skimming the homework to get a sense of students' knowledge of patterns. Refrain from helping students if they are having difficulty, however. Struggling with a problem or task is part of the learning process; it helps students to realize that they know more than they think.

After the students have had sufficient time (approximately 5 minutes) to work on the activity, facilitate a class discussion of Eratosthenes' Prime Sieve, comparing student results to the list of prime numbers already posted in the room. Ask students if they see any patterns in the list of the primes. Students will readily notice, for example, that 2 is the only even prime number; encourage them to make further conjectures about the spacing of the primes. Some students may guess that all odd numbers greater than 1 are prime, but the number 9 is a counterexample. As they conjecture, do not rush to correct them, but ask gentle,

Tips for Teachers

For many teachers it can be difficult to be patient after asking a question, and yet increasing wait-time beyond 3 seconds is positively correlated to improvements in student achievement and increases in the quality and amount of student contributions to classroom discussion (Cotton, 1988).

probing questions that encourage students to investigate possible patterns. Conclude the discussion by informing the class that prime numbers have been an object of human fascination since the time of the ancient Greeks. Prime numbers have surprising applications in fields as diverse as coded communication and Internet security.

Ask students to look at the essential questions posted in the room. Essential questions draw attention to the most important concepts of the unit and help to prevent lessons that present random assortments of facts. According to Heidi Hayes Jacobs (1997, p. 26), "An essential question is the heart of the curriculum. It is the essence of what you believe students should examine and know in the short time they have with you." An essential question is not designed to have a single right answer, but rather to be explored by students and teachers alike. As you introduce the essential questions, use them to call students' attention to two key words: *pattern* and *sequence*. Encourage students to come up with their own tentative definitions of these words. At the same time, ask students to complete a Quick Write in response to Essential Question 1: "When, how, and why are patterns useful in our everyday lives?" A Quick Write is a type of formative assessment that gives students time to explore what they think and know about a subject before being called upon to contribute to the discussion (Brewster & Klump, 2004). Asking students to respond to Essential Question 1 in writing early in the lesson encourages them to test their tentative definitions of *pattern* and invites them to connect the abstractions they learn about in the mathematics classroom to the concrete world. Students should write for no longer than 5 minutes; when time is up, collect the papers to review overnight. Students will return to them during the next class period.

The discussion of patterns should also serve as a springboard into the investigation of patterns through the Station Problems activity. Students will continue to work in prime clusters to complete the 5 stations that explore pattern seeking. Assign groups at random to each station. Have each group rotate to a new station every 5 minutes until everyone has had a chance to attempt them all. Students will be collaborating with the members of their groups to explore the connections between patterns and the world around them. They will build on their prior knowledge, explain their reasoning, and test their ideas.

Because students will be working together to solve the station problems, this is a good time for you to establish ground rules for group discussions in Algebra II. Rules encourage appropriate interactions such as active listening, a technique in which a listener repeats in his own words what the speaker just said. This requires listeners to be attentive and enables speakers to know whether they are expressing themselves clearly. Ground rules also discourage students who dominate conversations, show disrespect toward others, or speak out of turn. In this case, establishing ground rules also means assigning roles within the groups. To ensure that all students understand their roles while working in cooperative groups, write a

Tips for Teachers

In Algebra II, students will frequently be working in cooperative groups. In rigorous math courses—in which students conjecture, solve problems, and justify their responses using the language of the discipline—group collaboration is especially important. Effective cooperative groups have 5 essential elements: positive interdependence, face-to-face promotive interaction, individual and group accountability, interpersonal and small-group skills, and group processing (Johnson, Johnson, & Holubec, 1993):

- *Positive interdependence* means that all members of the group rely on each other. Each member of the group plays a unique role within the group.
- *Face-to-face promotive interaction* requires members to communicate, thereby strengthening their communication skills as well as helping them retain the information they are communicating.
- *Individual and group accountability* means that all students are responsible for their own and their group's learning. Randomly selecting students within the group to share the group's responses helps ensure that all students within the group are actively engaged in the tasks presented.
- *Interpersonal and small-group skills* are the tools for working together effectively with others. They are lifelong skills that many employers find essential.
- *Group processing* means that group members reflect on their performance and adjust it, as necessary, to maintain effectiveness.

description of each role on an overhead transparency or on the board while explaining the roles to the class:

- *Checkers* ensure that all group members understand or agree and that each member is ready to present if called upon.
- *Facilitators* read the questions to the group, focus the conversation on the current task, and encourage progress toward the next question.
- *Managers* pick up any materials needed for the activity and keep track of the time.
- *Recorders* take note of the group's responses, from ongoing discussion points through final consensus. (At the same time, each group member should also keep a personal record of the answers on his or her own copy of each station worksheet.)

Allow groups to decide who takes each role, but alert them that, over time, everyone will serve in each role. As groups work at the various stations, move from station to station, take note of individual contributions and group interactions, and provide guidance as necessary. Assess students' group work using the Group Participation and Collaboration Rubric (p. C-11).

Once all groups have worked at each station, facilitate a class discussion of results and address any questions that arise. Begin by choosing a group at random and ask its members to explain how they attempted to solve the first station problem. Give the other groups an opportunity to agree with, modify, correct, or expand upon the previous response. Then, repeat the process, asking students to address each station in turn. This strategy encourages students to remain actively engaged in discussion, to take responsibility for their learning, and to rely less on you for the correct answer.

Wrap up the day by using a formative assessment to gather information about students' learning thus far. Ask them to use sticky notes to record 3 things that they have learned so far, 2 things they do not fully understand, and 1 question they still have. This is called a 3-2-1 Assessment. It is typically used to understand students' thinking and to gauge their strengths and weaknesses. Divide a large poster board into three columns, labeling them *3*, *2*, and *1*. Ask students to stick each note in its corresponding space as they leave the classroom. Review the notes after class to identify frequently occurring questions or misconceptions. Use this information to help plan and adjust subsequent lessons.

Finally, distribute the Algebra Skills homework (pp. C-12–C-13), a short assessment of students' prior knowledge of integer arithmetic, solving linear equations and inequalities, and basic coordinate geometry. These are basic skills necessary to Algebra II and, specifically, to the work in this first unit.

Tips for Teachers

Randy Bomer (1995) recommends taking notes about students on a clipboard while in class. For each student, Bomer makes note of the situations in which he or she seems comfortable or uncomfortable, areas of knowledge he or she might bring to the class, reading or other extracurricular interests, anything the student says about school, or anything else that may help him know the student better. Even though these notes are imperfect and incomplete, they nevertheless provide a running history of the students' class experiences. Bomer explains to students that his notes are a form of valuing what they say. At the end of each week, Bomer places the notes into binders, one binder for each class. You may wish to keep your Algebra II notes in the same notebook where you have already placed students' autobiographies.

Day 3

Students explore arithmetic and geometric sequences and become familiar with the vocabulary for arithmetic sequences.

Materials & Resources

- ❑ Simon Says (p. D-2)
- ❑ Simon Says Key (p. D-3)
- ❑ Glossary (p. D-4)
- ❑ Class Journal Feedback Rubric (p. D-5)
- ❑ Patterns 1 transparency (p. D-6)
- ❑ Patterns 1 Key (p. D-7)
- ❑ Patterns 2 transparency (p. D-8)
- ❑ Patterns 2 Key (p. D-9)
- ❑ Simon Says 2 (p. D-10–D-11)
- ❑ Simon Says 2 Key (p. D-12)

To warm up, return students' Quick Writes in response to Essential Question 1. Ask students to read what they wrote and to reflect upon the patterns they learned about in the Stations activity. Give them 5 minutes more to write in response to their new understanding about patterns. Students should keep these reflections and insert them into their journals—which will be introduced later in the class period—as their first entries.

Provide the answers to the Algebra Skills homework either on the board or on the overhead projector and ask students to check their work. When they have finished, ask students to choose a few problems from the inventory to discuss. Ask students who solved the problems correctly to share their solutions, but also encourage students to share incorrect solutions—mistakes often provide more learning opportunities than correct solutions do. Make certain that all students know how to solve simple equations, perform integer arithmetic, and plot points before proceeding.

Next, distribute the Simon Says worksheet (p. D-2)—an introduction to arithmetic and geometric sequences—and engage students in a Think-Pair-Share activity (Lyman, 1981). Give students time to read the problem and individually complete the table for Michael's walk home. Then, have students share their results with a partner and work together to complete the table for Betty's walk. This particular strategy allows students to think about what they already know and to clarify their thinking by sharing with someone else. As students hear other viewpoints, they learn from each other's ideas and learning strategies. As students work, circulate through the room with your clipboard, observing and making notes as students grapple with these problems and make their own conjectures.

When students have finished, call on a few student pairs to share their findings with the class. Then, discuss the results with the entire class. Encourage students to look for patterns in the distance walked and total distance columns for both Michael and Betty. Without introducing formal terminology at this point, keep the following points in mind:

- The distance Michael walks in each segment produces an arithmetic sequence with common difference 0.
- The total distance Michael walks produces an arithmetic sequence with common difference 900.

- The distance Betty walks in each segment produces a geometric sequence with common ratio r.
- The total distance Betty walks after n segments may be represented as the sum of a finite geometric series.

Let students know that they will return to the Simon Says example again as their study of patterns in numbers continues.

Because new vocabulary will soon be introduced, this is a good time to address the need for a glossary and to explain the importance of maintaining an organization system for notes, homework, and important assignments. One suggestion for notebook organization follows:

- *Class Notes*: All notes taken in class and handouts should be kept in this section.
- *Glossary*: Defined vocabulary should be recorded here. (See the Glossary worksheet, p. D-4.)
- *Homework/Class Work*: Dated and graded assignments should be kept in this section.
- *Quizzes, Tests, and Projects*: Important formative and unit assessments should be kept here, as well as long-term projects.

Some teachers choose to conduct periodic notebook checks, awarding points to students who keep an organized notebook. Since Algebra II is a course that is generally taken by juniors and seniors and serves as a gateway to more advanced mathematics, frequently collecting and grading students' notebooks is probably less necessary than it is in earlier courses. Students should, however, be trying to identify organization strategies that work best for them and their particular styles of learning. Generally, the notebook should be checked after the first unit to ensure that students have established procedures and are following them. After that, random checks of notebooks will help students to remain organized and accountable for their learning.

Students should also keep a separate notebook for a math journal, in which they respond in writing to questions or problems. Until recently, journals have not been commonly used in mathematics classrooms. But research shows that journal writing helps students to recognize what they do and do not know, to connect current lessons to prior lessons, to reflect on and think critically about new ideas, and to keep their thoughts organized (Burchfield, Jorgensen, McDowell, & Rahn, 1993). Similarly, mathematics journals allow you to gain insight into students' mathematical thinking, to identify and address student misconceptions, and to assess students' study habits and attitudes (Rothstein & Rothstein, 2007, p. 22). Although students may at first resist the idea of journals, they typically come to see their value once they understand that the communication is about math and that you respond to their writing as a math teacher. As students develop the habit of writing, you can assess the effectiveness of daily lessons with the Class Journal Feedback Rubric (p. D-5)—adapted from Jim Burke's *Writing Reminders* (2003)—to provide feedback for their entries. Although students do not need to write in their math journals every day, they should write frequently enough to develop the habit of thinking and communicating about math in writing.

After explaining the class notebooks and the journals, display the Patterns 1 transparency (p. D-6) on the overhead projector and ask students to work together with their Think-Pair-Share partners to describe each pattern in words. Circulate through the classroom once again, observing, making notes, and—when students seem stuck—asking probing questions (e.g., "What is the relationship between the second number and the first?”). Once most students seem to be catching on to the patterns, call the class together for a general discussion of the results. Based upon your classroom observations, you might want to encourage students to take the lead in the discussion by asking questions that invite wider participation such as “What do the rest of you think?” or “Would anyone else like to add to that response?” You might also identify specific students—with their permission, of course—to come up with a thought-provoking question to help initiate or maintain the class discussion. Asking students to help facilitate discussion has many benefits, such as developing their speaking skills, boosting their confidence, and encouraging them to listen actively and to clarify each other’s thinking.

After describing the patterns on the first transparency, display the Patterns 2 transparency (p. D-8). In addition, distribute a hard copy to each student. Lead a class discussion that introduces and illustrates the vocabulary essential to the study of arithmetic sequences:

- *Sequence* (Question 1)
- *Term* (Question 1a)
- *Common difference* (Question 1f)
- *Arithmetic sequence* (Question 1g)

As each of the concepts arises, ask students to suggest appropriate terminology. Use Reflective Questioning to instigate their responses. Guide them to the word *sequence,* for example, by asking, “Where do you encounter ordered lists in your everyday life? What word would you use to describe an ordered list?” In each case, some students may already know (or be able to guess) the correct word or phrase; however, if no one volunteers it in a few minutes’ time, supply the proper vocabulary word and have students record it on the Glossary worksheet.

Students are likely to have particular difficulty remembering and working with the phrase *common difference*. The notion of working backward from a term to its predecessor can seem unnatural. Keep this possible difficulty in mind as you assess students’ understanding over the next several days. You may wish to use Hand Signals as a simple formative assessment tool:

- *Thumbs-up*: I get it!
- *Thumbs-to-the-side*: I’m not sure I get it!
- *Thumbs-down*: I don’t get it yet!

When the Patterns 2 transparency has been covered, students should file it in the “Notes” section of their notebooks.

As a wrap-up for Day 3, have the students work in pairs to complete the Glossary worksheet for the day’s vocabulary. Finally, distribute the Simon Says 2 homework (pp. D-10–D-11).

Day 4

Students work toward finding a formula for the n*th term of an arithmetic sequence and the sum of the first* n *terms of an arithmetic sequence.*

Materials & Resources

- ❑ Three Sequences transparency (p. E-2)
- ❑ Three Sequences Key (p. E-3)
- ❑ Movie Time (p. E-4)
- ❑ Movie Time Key (p. E-5)
- ❑ Summing It Up transparency (pp. E-6–E-7)
- ❑ Summing It Up Key (p. E-8)
- ❑ Even and Odd (pp. E-9–E-10)
- ❑ Even and Odd Key (p. E-11)

To warm up, place the Three Sequences transparency (p. E-2) on the overhead projector and have students work the prompts in their notebooks. The warm-up gives students more practice recognizing the basic properties of arithmetic sequences and determining those properties' values. When they have finished, students should pair up to discuss their answers. Ask volunteers to share responses to the Three Sequences transparency with the class. As they share, prompt the students to use correct vocabulary: each of the examples is an arithmetic sequence; t_1 and t_7 denote the 1st and 7th terms of the sequence, respectively; d denotes the common difference of a given arithmetic sequence. Prompt different individual students to define each term as it is needed. Requiring students to express not only the definitions in their own words but also in the language of mathematics focuses students' thinking and reinforces the importance of precise terminology in your classroom.

After the warm-up, review the Simon Says 2 homework. Encourage students to continue to speak precisely using the language of mathematics as they discuss the answers.

Next, form groups of four students and distribute one Movie Time worksheet (p. E-4) to each student for small-group discussion. Groups can be created in a variety of ways:

- When students have recently worked in pairs, have each pair join with another nearby.
- Have students return to their prime clusters from Days 1 and 2.
- If space permits, have students line up from tallest to shortest. Group the two tallest students with the two shortest, the next tallest students with the next shortest students, and so on.
- Have students line up in order of their birthdays. Group the two youngest students with the two oldest, and so on.
- Prepare an ordinary deck of cards so that the number of cards equals the number of students in the class. Have each student select a card. Create groups according to the face value on the cards (e.g., jacks, aces, 7s).

If the number of students in your class is not a multiple of four, adapt these strategies to allow a few groups of three.

When forming groups, remember to keep in mind students' gender, ability, experience, and needs as well as your purpose for assigning groups. Random groups are fine for temporary work; however, thought and care should be given for long-term group work. In addition, to facilitate students' learning from each other, take time to reformulate groups throughout the unit or course so students have opportunities to work with many different people (Center for Teaching Excellence, 2004).

As groups discuss the Movie Time worksheet, circulate through the room with your clipboard, listening and observing, noting strategies students use and misconceptions they have. Resist the temptation to interject or to give away the answers; instead, pose questions that help students think through their conjectures. Many (perhaps most) will attempt to calculate the number of seats in the theater by determining the number of seats in each row and then adding the 20 terms. If calculators are available, they may perform all the necessary arithmetic without writing anything. However, encourage students to record the steps they have taken in a way that would make the process clear to someone who does not have a calculator. Challenge groups to find shortcuts that simplify the problem.

When groups have finished, discuss the results. Ask groups to share solution strategies with the class; encourage students to show their work on the board or overhead projector. The primary goal of the activity is for students to identify patterns in the data, to make conjectures about those patterns, and to refine their conjectures by testing them. Throughout the discussion, emphasize that testing conjectures is one of the best ways to learn. In addition, emphasize the real-world importance of the problem. For example, it would be time consuming to count every row in the theater. Knowing how to quantify the number of seats quickly would allow the party planners to move on to solving the real problem—what to do with all the extra prom attendees.

One way of helping students test their conjectures is to challenge the class to find a formula for the number of seats in the nth row of the theater when n is an integer and $1 \leq n \leq 20$. If the class has not already done so, define a sequence by setting t_n equal to the number of seats in the nth row, where n is a positive integer. Invite the students to observe patterns in the values of t_n. Students should recognize, in particular, that this is an arithmetic sequence with first term $t_1 = 15$ and common difference of 2. Mathematicians express this relationship with the recursive formula

$$t_1 = 15$$
$$t_{n+1} = t_n + 2 \text{ for } 1 \leq n \leq 19.$$

However, very few students will be ready to deal with this symbolic formulation yet.

Point out that, while it was easy to figure out how many seats were in the 3rd and 7th rows by repeatedly adding the common difference, the higher rows are more difficult; doing the calculation for a large theater would be tedious and would increase the risk of propagating arithmetic errors. Pose the question, "Suppose you were designing a similar theater with 50 rows and you wanted to know how many seats to place in Row 47. Would you want to extend the sequence that far just to do the calculation?"

Encourage students to carefully enumerate the steps required to calculate the first few terms of the sequence:

1. $t_1 = 15$
2. $t_2 = 15 + 2$
3. $t_3 = t_2 + 2 = 15 + 2 + 2 = 15 + 2(2)$
4. $t_4 = t_3 + 2 = 15 + 2(2) + 2 = 15 + 2(3)$

Tips for Teachers

Many performance venues have websites where students can investigate the actual arrangement of theater seating. One of the best of these is maintained by the Auditorium Theatre of Roosevelt University (2006) in Chicago. As in many other venues, the Auditorium features several tiers of balcony seating in addition to 37 rows of seats on the main floor.

Focusing upon arithmetic steps rather than numerical values in this way helps to clarify the process of finding the number of seats in any given row. Challenge students to write a formula for the number of seats in Row *n*. Before moving on, ask cueing questions that lead students to conclude that, before a general formula can be accepted, it must be tested against the data to see whether it works. Ask volunteers to present their conjectures. Ask the class to test each conjecture for several different values of *n*.

Through this process of conjecture and testing, students are likely to agree on the formula $t_n = 13 + 2n$, though the equivalent formula $t_n = 15 + 2(n - 1)$ corresponds most directly to the steps they have taken. Problems 1 and 2 on the Movie Time worksheet are easily solved using either formula. Invite students to apply the formulas to the 47th and 50th rows of the similar theater that was proposed earlier. They should see that $t_{47} = 15 + 2(46) = 107$ and $t_{50} = 15 + 2(49) = 113$. Inviting students to evaluate whether 113 seats is a reasonable number of seats for the back row of a theater reemphasizes the real-world applications of the math.

Tips for Teachers

Legend has it that, at age 10, the German mathematician Carl Friedrich Gauss (1777–1855) discovered (or invented) the folding method for summing a sequence with an even number of terms when his teacher asked his class to sum the natural numbers from 1 to 100. As Howard Eves (1990, p. 476) relates, "Carl alone had the correct answer, 5050, but with no accompanying calculation. Carl had mentally summed the arithmetic progression 1 + 2 + 3 + . . . + 98 + 99 + 100 by noting that 100 + 1 = 101, 99 + 2 = 101, 98 + 3 = 101, and so on for fifty such pairs, whence the answer is 50 × 101, or 5050."

Next, explore with the class how the formula for t_n can be used to answer Questions 3 and 4 on the Movie Time worksheet. During the exploration, you may learn that, to answer Question 4, some students paired Rows 1 and 20, Rows 2 and 19, Rows 3 and 18, and so forth using a folding method, producing 10 pairs of 68 seats each to discover that the total number of seats in the theater is 10 × 68 = 680. If so, the Summing It Up transparency (pp. E-6–E-7) introduces students to a modified version of the folding method that produces an expression for 2S, where Part 1 demonstrates that S is the sum of the first 4 terms (as in Problem 3) and Part 2 the first 20 terms (as in Problem 4) using the formula for the *n*th term of the Movie Time sequence. In order to encourage students to engage each other in the discussion, use the Think-Pair-Share collaborative discourse strategy: Tell students to take a few minutes to read and think about the problem, then have them discuss it with a nearby partner. After a few minutes more, ask pairs to share their ideas about this new approach to summing the sequence with the class.

One advantage of the doubling method over the folding method is that the doubling method may be used with a sequence of any length, while the folding method relies on pairing up an even number of terms. It must be modified to be used when there is an odd number of terms. Help students explore the difference between the methods by proposing a pair of thought experiments. First, ask students to apply the folding method to calculate the sum of the first 5 terms of the Movie Time sequence. The folding pattern pairs up t_1 with t_5 and t_2 with t_4, but it leaves t_3 without a partner. The sum S of the first 5 terms of the sequence is:

$$\begin{aligned} S &= (t_1 + t_5) + (t_2 + t_4) + t_3 \\ &= [15 + (15 + 8)] + [(15 + 2) + (15 + 6)] + (15 + 4) \\ &= 38 + 38 + 19 \\ &= 2(38) + 19 \\ &= 95 \end{aligned}$$

Next, have them apply the doubling method. While there are more individual sums to compute, there is no unpaired term:

$$\begin{aligned} 2S &= (t_1 + t_5) + (t_2 + t_4) + (t_3 + t_3) + (t_4 + t_2) + (t_5 + t_1) \\ &= 38 + 38 + 38 + 38 + 38 \\ &= 5(38) \\ &= 190 \end{aligned}$$

Dividing by 2 produces the same result as above: $S = 95$.

By now, students should begin to observe patterns in the various Movie Time examples. In each example, the sums of paired terms are constant. Using the folding method with an odd number of terms, the unpaired term has a value equal to exactly one-half of the sum of any two paired terms. Encourage students to identify other patterns in the sums. Some students may have already determined the formula for the sum S_n of the first n terms:

$$S_n = \frac{n}{2}(t_1 + t_n) = \frac{n}{2}[30 + 20(n - 1)]$$

As you study patterns further, take time to discuss the various uses of subscript notation. Until now, students have been using the symbol t_n to denote the nth term of a sequence, but when discussing more than one sequence at a time, it is helpful to have a variety of notations available. Lowercase letters with subscripts are normally used to denote the terms of a sequence. In addition to t_n, the symbols a_n, b_n, and c_n are also commonly used. Capital-letter notation is typically reserved for sums of a sequence. With any given sequence, the notation S_n is frequently used to denote the sum of the first n terms.

As a wrap-up, have students return to their groups to complete the first problem on the Even and Odd homework (pp. E-9–E-10), which provides practice using the formulas for sequences. Like the Movie Time sequences, the problems on the homework are arithmetic sequences with common difference $d = 2$. They should help students both to build confidence in determining the formula for the nth term and to conjecture a general formula for the sum of the first n terms of an arithmetic sequence. As students work, circulate through the classroom once more, observing, taking notes, and asking reflective questions. Always remember that your goal is to nudge students toward deep understanding of the concept of sequences and their sums. Students must learn to rely upon their own abilities to reason logically and to apply what they have so far learned to new situations. Students should complete the rest of the homework before the next day's class.

Tips for Teachers

As an aid for reflective questioning, you may want to keep a set of questions at hand. For example, write the following questions on index cards, one question per card.

- Why do you think that?
- How do you know?
- Could you give me an example?
- What data do you have to support your position?
- Tell me more about . . .
- How might you find out or confirm?
- What essential question might this topic refer to?
- What more would you like to know about . . . ?
- Does this situation remind you of other problems you have solved?
- Can you find a similar, but simpler, problem to help you understand this one?
- How would you explain so that a 6-year old could understand that concept?
- Is there another way to represent this information?
- Can you solve the problem in another way?
- What is the main idea of . . . ?
- How does . . . affect . . . ?
- What is a new example of . . . ?
- How are . . . and . . . similar and how are they different?

Over time, add to your repertoire by inserting new questions, revising old ones, and constantly shuffling the deck so that you never settle into predictable habits.

Day 5

Students develop and apply formulas for the n*th term of an arithmetic sequence and the sum of the first* n *terms of an arithmetic sequence.*

Materials & Resources

- ❑ Summing It Up Again Transparency (pp. F-2–F-3)
- ❑ Summing It Up Again Key (p. F-4)
- ❑ Arithmetic Sequences and Series transparency (p. F-5)
- ❑ Arithmetic Sequences and Series Key (p. F-6)
- ❑ Arithmetic Sequences and Series Graphic Organizer (p. F-7)
- ❑ Arithmetic Sequences and Series Graphic Organizer Key (p. F-8)
- ❑ Arithmetic Sequences and Series Practice (p. F-9)
- ❑ Arithmetic Sequences and Series Practice Key (p. F-10)

To warm up, have students check their homework with a nearby partner. Circulate through the room with your clipboard, meanwhile taking attendance and observing students' interactions. Take note of students who are having difficulty with basic calculations as well as those who have made progress on the conjectures in Problems 1d, 2d, and 3.

Much of Day 5 will be spent reinforcing and extending students' understanding of arithmetic patterns and the generalized formulas that students explored on the Even and Odd homework. Take advantage of your notes as the class progresses. Try re-pairing students, placing those who have a solid grasp of the concepts with those who are struggling. Encourage students to learn from each other. Encourage them also to learn from each specific case they are discussing and to identify features the cases have in common. The latter is an important feature of inductive learning called concept formation. As students categorize data—identifying common differences and the first and last terms of a sequence, for example—they learn how to recognize patterns abstractly and to develop more sophisticated, testable generalizations about the world.

When pairs have reviewed their homework, conduct a brief class discussion of Problems 1 and 2. Connect prior knowledge of the even and odd natural number sequences to new concepts and terminology. In the discussion of Problem 1, for example, ask what patterns they have observed in the even numbers in previous courses. Some students might mention that the even numbers are equally spaced; help them to connect this idea to the fact that the common difference, d, is equal to 2.

In Problems 1c and 2c, students practiced determining a formula for the nth term of a sequence. Because some students may have found the problems difficult, in addition to asking those who succeeded to share their solution strategies with others, encourage students to use the nth term formula in 1c as a stepping stone to 2c. Invite them to look for similarities between the two sequences. This suggestion alone may be sufficient to spark students' understanding; if not, lead them step-by-step through the reasoning until they take over. They may notice that $b_1 = a_1 - 1$, and that the sequence $b_1, b_2, \ldots$ has the same common difference as $a_1, a_2, \ldots$, so that $b_n = a_n - 1$ and thus $b_n = 2n - 1$. Problems 1d and 2d offer additional practice with the doubling method for summing a sequence. If by now this process seems repetitive to students, then they will be ready to move beyond specific examples to explore the process in a general setting.

The Summing It Up Again transparency (pp. F-2–F-3) works through the process of solving Problem 3 from the Even and Odd homework. It also serves as a springboard to discussion of general formulas related to arithmetic sequences and series. Display Part 1 of the Summing It Up Again transparency and walk the class through the calculation of the first 4 terms of a 10-term arithmetic sequence with 1st term c_1, common difference d, and nth term c_n. Then, ask students to pair up again to calculate the remaining terms and to conjecture a general formula for c_n. (Again, try to pair up students who solved or came close to solving the problem with those who struggled.) Observe pairs as they work, taking special note of those making good progress toward the general formula.

When pairs have had sufficient time to complete their calculations, ask for volunteers to share their results, including conjectures as to the general form of c_n. Call on pairs who have come closest to finding a correct formula to demonstrate and explain their reasoning. If no one has arrived at a final expression for c_n, guide students to develop a formula by asking what the pattern of the calculations suggests. Students should test and refine each formula until the class agrees that it is a successful generalization of the sum of an arithmetic series.

Then, display the second page of the Summing It Up Again transparency. It addresses the problem of calculating the sum S_{10} produced by adding together the 10 terms of the sequence. Ask students to think-pair-share the steps in the calculation of S_{10}.

Students should be ready for a more general exploration of arithmetic sequences and series. To facilitate this exploration, display the Arithmetic Sequences and Series transparency (p. F-5), but conceal the bottom portion of the transparency. Conduct a class discussion of the first section, "Sequences and Series in General." Initially, students may be uncomfortable with this abstract notation, so be sure to supplement the discussion with plenty of old and new examples, such as the prime number sequence, the Simon Says and Movie Time sequences from Day 3, the three sequences from Day 4, and the even and odd number sequences from the Even and Odd homework. Before moving on to the second section of the transparency, allow time for students to respond to the question, "What is the difference between a sequence and a series?" Encourage students to define the terms *sequence* and *series* carefully, using the language of mathematics, and to develop a consensus response to the question—that is, a response everyone agrees with.

Students should continue working in pairs on the questions in the second section of the Arithmetic Sequences and Series transparency. As they work, take notes and respond to questions without giving away the answers. For example, when faced with the task of doing a purely symbolic, "numberless" calculation, a student may vent his or her frustration, exclaiming, "I don't know how to do this!" Respond with encouragement: "Break the problem down. Where do you think you should start?" It may help to ask the student to recall specific numerical examples in which the doubling method for calculating the sum of an

Tips for Teachers

If time permits, share the following anecdote with the class, or display it on a transparency or bulletin board:

> The American mathematician Antoni Zygmund (1900–1992) came to the United States as a refugee from German-occupied Poland at the start of World War II. As a professor at the University of Chicago, he devoted much of his career to studying and teaching about the properties of sequences and series. He was constantly encountering new aspects of popular culture, which he viewed through the lens of mathematics. One October, when asked for his impressions of baseball's World Series, he replied, "I think it should be called the World Sequence." (Coifman & Strichartz, 1989, p. 348)

arithmetic series has been used. The patterns in these specific examples should serve as a guide to the general calculation.

When pairs have had sufficient time to grapple with these questions, bring the class together again to discuss the results. Encourage students to recognize two related points:

1. The expression $2t_1 + (k - 1)d$, which appears in the equations for the sum of paired terms and the sum of the series, is equal to $2t_1 + d(k - 1)$ by the commutative property of multiplication.
2. The type of problem being solved determines which form of the expression to use. When k values vary, choose the first expression; when d values vary, choose the second.

To wrap up and summarize the day's instruction, introduce the Arithmetic Sequences and Series Graphic Organizer worksheet (p. F-7). Graphic organizers visually represent information that students are learning, familiarizing them with new concepts and the relationships between them (Marzano, Norford, Paynter, Pickering, & Gaddy, 2001, pp. 284–286). This graphic organizer serves as a summary of what students have learned so far. Before class ends, check that all graphic organizers have been completed correctly. Pair students whose graphic organizers reveal misunderstandings of the concepts with others who appear to grasp the concepts well. Encourage them to compare their work. Students should file complete and correct graphic organizers in the "Notes" section of their notebooks for ready reference.

While the graphic organizer provides students with a way to visualize the lesson, it is important that they practice using the concepts and formulas they have learned. Practice should ask them to demonstrate their skills in several ways:

- Finding the value of various terms of an arithmetic sequence
- Determining the place of a term (the term number) in an arithmetic sequence or series
- Finding the common difference given a general rule and one of the terms
- Finding the sums of finite arithmetic series

With this in mind, wrap up the day's instruction by distributing the Arithmetic Sequences and Series Practice homework (p. F-9). As they complete the homework, encourage them to refer to their graphic organizers as needed.

Day 6

Students establish connections between arithmetic sequences and linear equations. They use those connections to explore geometric sequences and series. Sigma notation is introduced as convenient shorthand for series.

Materials & Resources

- ❑ Math journals*
- ❑ Lining Up (p. G-2)
- ❑ Lining Up Key (p. G-3)
- ❑ Butcher paper*
- ❑ Calculators*
- ❑ Graph paper*
- ❑ Growing Geometrically (p. G-4)
- ❑ Growing Geometrically Key (p. G-5)
- ❑ Shrinking Geometrically (p. G-6)
- ❑ Shrinking Geometrically Key (p. G-7)
- ❑ Sigma Notation (p. G-8)
- ❑ Sigma Notation Key (p. G-9)
- ❑ Geometric Sequences Vocabulary transparency (p. G-10)
- ❑ Geometric Sequences Vocabulary Key (p. G-11)

* Materials or resources not included in published unit

Because Day 6 extends students' understanding of patterns by introducing geometric sequences and series, warm up by asking students to write in their journals in response to the following prompt:

> Biologist Leonard Hayflick discovered that a human cell cultured in a petri dish can divide up to 50 generations, but no more. Assume that Hayflick's petri dish contains 1 cell. That cell splits into 2 cells, creating a 2nd generation. Each of those 2 cells splits into 2 cells, creating 4 total cells in the 3rd generation, and so on, for 50 generations. Does this cell division form a pattern? Why or why not?

After students have written for 3 minutes, ask volunteers to read their work aloud, and allow students to speculate about both their peers' work and the prompt. They will return to their journals at the end of class to generalize from the prompt and respond to Essential Question 2.

Next, ask students to pair up with someone they have never before worked with. Together, they should complete the Lining Up worksheet (p. G-2), which establishes the connection between arithmetic sequences and equations of lines. Problem 5 requires a calculator and may be skipped if calculators are unavailable. When students have finished, lead a brief discussion of the results.

During the discussion, allow plenty of time for students to make and test conjectures, as you did in the warm-up. Resist the temptation to step in with the correct answer; rather, ask questions that encourage students to connect arithmetic sequences to concepts from Algebra I. In particular, ask how the results of Problem 4 can be generalized to any arithmetic sequence with 1st term t_1 and common difference d. The goal is for students to recognize that the points of an arithmetic sequence lie along a line having slope d and y-intercept $(t_1 - d)$. To guide students in making these connections, ask questions that require them to draw on prior knowledge from Algebra I: "If you look at two neighboring points on the graph of the sequence, what is the rise, and what is the run?" Students should see that the rise is equal to d while the run is equal to 1, creating an explicit link between the slope of the line and the common difference d.

In order to connect students' prior knowledge to what they have learned so far, you might also ask the following question: "What is the relationship between the y-intercept and the terms of the sequence?" This is an example for which the nth term formula is especially useful: if the sequence is extended backward, the y-intercept corresponds to the value of the "zeroth" term of the sequence:

$$t_0 = t_1 + (0 - 1)d = t_1 - d$$

Be on the lookout for potential difficulties in working with the nth term formula. If students use the calculation of t_0 in which d appears on the left, they may be led to the erroneous calculation:

$$t_0 = t_1 + d(0 - 1) = t_1 + d - 1$$

Similarly, students may be prone to errors while using the version in which d appears on the right:

$$t_n = t_1 + (n - 1)d$$

Another potential problem: when $t_1 = 21$ and $d = -5$, students may forget to enclose –5 in parentheses, producing

$$t_n = 21 + (n - 1) - 5, \text{ or}$$
$$t_n = n + 15,$$

which is incorrect. Without telling them explicitly, help students to conjecture, test, and finally realize that, if they use the version of the equation with d on the left

$$t_n = t_1 + d(n - 1)$$

with the correct values of t_1 and d, they are much more likely to come up with the correct linear equation.

As you discuss the Lining Up worksheet with the class, introduce the term *linear growth* to describe the growth pattern of arithmetic sequences. Use this as a springboard to introduce the idea of *geometric* (also known as *exponential*) *growth*. A *geometric sequence* is one in which each successive term is produced by multiplying the preceding term by a constant factor r, called the *common ratio*. Just as the graph of an arithmetic sequence produces points on a line, the graph of a geometric sequence produces points on an exponential curve—that is, the graph of a constant multiple of $y = r^x$.

The introduction of geometric growth previews a lot of new and important information. Exponential functions and their graphs will be studied in detail later in Algebra II. Therefore, be sure to illustrate the ideas with concrete examples. Many students will be familiar with e-mail chain letters and other forms of spam. Ask volunteers to suggest other examples, and list their real-world examples on the board or butcher paper. In addition to e-mail chain letters and spam, the list may include investment growth,

Tips for Teachers

To supply students with more real-world examples of geometric growth and decay, check out any of the following excellent and readily accessible resources:

- Lottery winnings: Mathematical Science Education Board (MSEB) (1998, pp. 111–114); Wattenberg (1997c)
- Car loans and leases: Wattenberg (1997b)
- Consumer debt: MSEB (1998, pp. 87–90)
- Elimination of medication from the body (an example of geometric/exponential decay): MSEB (1998, pp. 80-82); NCTM (2000, pp. 303–5)
- Carbon dating and radioactive decay: Maor (1991, pp. 31–32), Wattenberg (1997a)
- Population growth: Morrow (1999)

population growth, Internet expansion, and more. Refer to these examples frequently throughout the remainder of the unit.

Employ the Think-Pair-Share strategy as students complete the Growing Geometrically worksheet (p. G-4), which formally introduces geometric sequences by exploring an e-mail chain-letter scenario in detail. Ask students to graph the terms of the sequence as they did on the Lining Up worksheet, then ask them to compare the rates of growth in each example. As they share their results in a class discussion, ask students to contrast linear and geometric growth rates. Encourage the class to consider other consequences of unchecked geometric growth in the real world in addition to the consequences of computer storage space, which the worksheet illustrates.

Next, have students work in groups of four on the Shrinking Geometrically worksheet (p. G-6), which uses Betty's walk from the Day 3 Simon Says worksheet to illustrate the related concept of *geometric* (or *exponential*) *decay*. Many students will be unfamiliar with the word *decay*, except possibly in the context of tooth decay. The dental analogy may be helpful. Just as tooth decay can cause tooth enamel to disappear rapidly, exponential decay describes the rapid decrease of a numerical sequence.

Students will probably need some guidance in using proper notation, so offer encouragement as they work. Whenever possible, ask other students to suggest solutions to problems. As you have done throughout the unit, help students break down problems into smaller, more easily comprehensible units. Remind them of previous problems which had comparable solutions, and help them to discover new solution strategies that they can rely on to solve the problems themselves. When groups have completed their work, ask for volunteers to explain their results, and then ask students to offer other examples of geometric shrinking (e.g., radioactive decay, other examples from physics and chemistry if students have taken these courses).

Tips for Teachers

Betty's apparent failure to get home while playing Simon Says is a modern example of Zeno's Paradox of Achilles and the Tortoise, a logical paradox first described by the Greek philosopher Zeno of Elea in the 5th century B.C. Isaac Reed (1998) offers an excellent discussion of Zeno's paradox and other historical problems in mathematics at the website of the Math Forum at Drexel University.

In the discussion of the Shrinking Geometrically worksheet, students may have realized that writing long sums such as

$$\left(\tfrac{1}{2}\right)^1(5{,}280) + \left(\tfrac{1}{2}\right)^2(5{,}280) + \left(\tfrac{1}{2}\right)^3(5{,}280) + \ldots + \left(\tfrac{1}{2}\right)^9(5{,}280)$$

can be tedious. Distribute the Sigma Notation worksheet (p. G-8), which previews the notation through examples of both geometric and arithmetic series. Students should discuss the worksheet in their groups and begin working the first two problems. The rest of the problems should be assigned as homework.

Next, use the Geometric Sequences Vocabulary transparency (p. G-10) to lead a class discussion that introduces and illustrates several instances of the terms *common ratio* and *geometric sequence*. Many students will be readily able to guess the term *geometric sequence*. Encourage them to reason by analogy from the concept of common difference to come up with *common ratio* (or something close to it). Before the end of the discussion, take the opportunity to introduce the term *geometric series* to refer to the sum of the terms of a geometric sequence. Instruct students to write new vocabulary terms and their definitions into their glossaries. The following terms were introduced today:

- Linear growth
- Geometric growth
- Exponential growth
- Geometric sequence
- Common ratio
- Geometric series
- Sigma notation

Wrap up Day 6 with a return to students' journal entries. Ask them to reread their writing and respond to it in light of what they have learned about geometric sequences and series. In addition, ask them to write a journal entry in response to Essential Question 2: "How do you know when an observed pattern forms a sequence?" By now, students should recognize that a sequence is an ordered list of numbers. While students might suggest other sequences, the sequences in this unit either increase or decrease according to a common difference or common ratio. Your review of their responses should reveal how well students understand the concepts introduced so far. If the journals reveal significant misconceptions, adjust your instruction of the class to emphasize the similarities among sequences or identify students who may benefit from individual tutoring. Finally, if time is short, have students finish the new entries as homework.

Day 7

Students explore the properties of sigma notation and develop techniques for calculating sums of geometric series.

Materials & Resources

- ❑ Half and Half Again transparency (p. H-2)
- ❑ Half and Half Again Key (p. H-3)
- ❑ Summing Geometrically (p. H-4)
- ❑ Summing Geometrically Key (p. H-5)
- ❑ Calculators*
- ❑ Index Cards*

* Materials or resources not included in published unit

Ask students to pair up with a nearby partner and warm up by checking each other's work on the Sigma Notation worksheet. Emphasize that they should try to reach a consensus on the solutions to the problems. Circulate through the room with your clipboard, taking attendance and assessing student understanding of the symbolic representations of series. In addition, collect students' journals so you can review their responses to Essential Question 2 after class.

When pairs have finished checking their work, discuss the homework. Begin by reviewing the basics of sigma notation: If $t_1, t_2, \ldots t_n, \ldots$ is a sequence of at least n terms where $1 \leq k \leq n$, and t_k is the kth term in the sequence, then the sum $t_1 + t_2 + \ldots + t_{n-1} + t_n$ of the first n terms of the sequence is equal to:

$$\sum_{k=1}^{n} t_k$$

Reading the expression aloud, one would say "the sum of t_k from k equals 1 to k equals n" or "the sum of t_k as k ranges from 1 to n."

Invite pairs of students to come to the board or to the overhead projector and demonstrate their work on each of the homework problems. In each case, ask them to decide whether the series is arithmetic, geometric, or neither. Series 1, 2, and 4 are arithmetic, and in each case determining the sum should have been fairly easy. Series 3 is geometric, but because it is a sum of distinct powers of 10, students should still have been able to calculate the sum using the properties of decimal notation.

This is a good time to point out a few rules for sigma notation that result from the properties of the real numbers. Before proceeding, ask volunteers to state and provide examples of each of the following properties:

- Commutative property of addition
- Associative property of addition
- Commutative property of multiplication
- Associative property of multiplication
- Distributive property of multiplication over addition

When students seem comfortable with these real-number properties, discuss their consequences for sigma notation. In particular, when n is a positive integer and a and b are constant,

$$\sum_{k=1}^{n} (a t_k + b) = a\left(\sum_{k=1}^{n} t_k\right) + \sum_{k=1}^{n} b = a\left(\sum_{k=1}^{n} t_k\right) + nb$$

In other terms:

$$\sum_{k=1}^{n}\left(at_k + b\right) \neq \sum_{k=1}^{n} at_k + b$$

Students often have trouble with these two representations, so take a few minutes to introduce the Pair-and-Compare technique. Ask students to turn to their neighbors and compare notes on the rules of sigma notation. They should add to, correct, or clarify their notes, as needed.

Students may find it more helpful to discuss sigma notation using specific values for *a*, *b*, and *n*. For example, ask them to carry out the calculations that will show the following is true:

$$\sum_{k=1}^{5}(2k+3) \neq \sum_{k=1}^{5} 2k + 3$$

They should recognize that the series on the left is an arithmetic series of 5 terms with 1st term equal to 5 and 5th term equal to 13. Its sum is $\frac{5}{2}(5+13) = 45.$ They should also see that the series on the right is equal to 3 plus an arithmetic series of 5 terms, with 1st term equal to 2 and 5th term equal to 10. The value of the expression is $3 + \frac{5}{2}(2+10) = 33$.

Another useful illustration of the rules of sigma notation and of geometric series is Betty's walk from the Simon Says worksheet. Betty's walk produces a 9-term geometric sequence with 1st term t_1 = 2,640 and common ratio $\frac{1}{2}$. Ask volunteers to come to the board or overhead projector to develop conjectures for an expression for the *k*th term of the sequence (for $1 \leq k \leq 9$). The rest of the class should test the volunteers' reasoning and offer constructive ideas. Through calculations such as

$$t_2 = \frac{1}{2}t_1$$

$$t_3 = \frac{1}{2}t_2 = \frac{1}{2}\left(\frac{1}{2}t_1\right) = \left(\frac{1}{2}\right)^2 t_1$$

$$t_4 = \frac{1}{2}t_3 = \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 t_1\right] = \left(\frac{1}{2}\right)^3 t_1$$

some students may conjecture:

$$t_k = \left(\frac{1}{2}\right)^{k-1} t_1 = \left(\frac{1}{2}\right)^{k-1}(2,640)$$

In response, ask, "For which values of *k* is this formula valid?" If students volunteer the range of values $2 \leq k \leq 9$, then reply, "What does this formula tell you when $k = 1$?" (Take the opportunity to review the definition of the zeroth power of a nonzero number, namely, that $x^0 = 1$ when $x \neq 0$.)

Finally, while discussing Betty's walk, be prepared in case students object that the previous day's work suggested a different formula:

$$t_k = \left(\frac{1}{2}\right)^k (5,280)$$

In that case, invite students to substitute several values of *k* to show that the formulas are equivalent.

The Half and Half Again transparency (p. H-2) revisits Betty's walk and continues to explore techniques for computing sums of geometric series. Have students form groups of 4 to discuss strategies for summing the series of powers of $\frac{1}{2}$. If calculators are available, allow students to use them. When groups are finished, ask them to share their ideas. Help lead them through the calculations to determine that:

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k-1} = 2-\left(\frac{1}{2}\right)^{n-1} = \frac{2^n-1}{2^{n-1}}$$

Consequently:

$$\sum_{k=1}^{n}\left[\left(\frac{1}{2}\right)^{k-1}(2,640)\right] = 2,640\left[2-\left(\frac{1}{2}\right)^{n-1}\right] = 2,640\left(\frac{2^n-1}{2^{n-1}}\right)$$

The last expression represents the distance Betty has walked by the end of segment n. Ask students to substitute the values 1–9 for n in the formula, and compare the results to those tabulated for Betty on the Simon Says worksheet. Students should observe that the results produced by the formula match those given in the table.

Use the Summing Geometrically worksheet (p. H-4) to introduce more general methods of summing geometric series. Students should remain in their groups, but this time they should work the problems without calculators. The worksheet develops an argument with the same solution as the formulas on the Half and Half Again worksheet, but with more symbolic manipulation than numerical calculation. This repetition should help solidify students' understanding of the formulas. There will probably not be sufficient time to complete the worksheet in class; assign anything unfinished as homework.

To wrap up, ask students to complete an Index Card assessment. Like the 3-2-1 Assessment and the journal assignment from Day 6, it provides valuable information about each student's current level of understanding. Distribute an index card to each student. On one side of the card, students should summarize their understanding of the concepts presented thus far. On the other side, they should identify something about the unit or today's learning that they do not yet fully understand—it can be written as a statement or a question. Students should turn in the cards before leaving class. Prior to the next day's lesson, review students' journals and the index cards. Adjust future instruction to acknowledge common understanding and to address common problems.

Day 8

Students learn a general method for computing sums of geometric series, review concepts and terminology related to geometric sequences and series, and use these concepts to solve problems.

Materials & Resources

- ❑ General Geometric Series transparency (p. I-2)
- ❑ General Geometric Series Key (p. I-3)
- ❑ Geometric Sequences and Series Graphic Organizer (p. I-4)
- ❑ Geometric Sequences and Series Graphic Organizer Key (p. I-5)
- ❑ Geometric Sequences and Series Practice (p. I-6)
- ❑ Geometric Sequences and Series Practice Key (p. I-7)

To warm up, have students return to their Day 7 groups to discuss the answers to the Summing Geometrically worksheet. As in previous days, ask volunteers to present their solutions, then discuss the results with the class. By now, sharing in this manner should be a classroom routine that invites students to explore and develop key concepts together and to practice communicating with the language of mathematics.

To bridge previous learning to a discussion of general geometric sequences and series, ask a volunteer to write on the board definitions for geometric sequences and series from his or her glossary. As the volunteer writes, the rest of the class should offer suggestions and clarify vague terms. They should know by now that a geometric sequence is an ordered list of terms $t_1, t_2, t_3, \ldots t_n$ where the ratio

$$\frac{t_{k+1}}{t_k}$$

of the $(k + 1)$st term to the kth term has the constant value r, called the common ratio.

Remind students that, when exploring Betty's walk, they combined the equations $t_1 = 2{,}640$ and $r = \frac{1}{2}$ to produce a formula for the kth term:

$$t_k = 2{,}640\left(\frac{1}{2}\right)^{k-1}$$

Ask students to express this formula symbolically in terms of the 1st term t_1 and the common ratio r. Some students may notice immediately that the kth term for Betty's walk has the form $r^{k-1}t_1$. Nevertheless, be patient as all students work toward a consensus on this formula. If necessary, pair students up to show why the general formula works. Proceeding inductively as before, they should discover patterns:

$$t_2 = rt_1$$
$$t_3 = rt_2 = r(rt_1) = r^2t_1$$
$$t_4 = rt_3 = r^3t_1$$

They should also realize that $t_k = r^{k-1}t_1$ (or, by the commutative property, t_1r^{k-1}).

Having derived a formula for the kth term of a geometric sequence (or series) in this specific case, students should now be ready for a discussion of the general case. If students need additional practice with specific cases, however, return to other problems from previous days' work. Inductive teaching can sometimes seem repetitive, but keep in mind that students are not only developing deep understanding of sequences and series, they are also learning complex abstractions and symbolic representations that enable them to generalize from real-world events. Giving students time to create generalizations on their own takes patience, but it pays off in their comprehension of mathematics.

Display the General Geometric Series transparency (p. I-2) and lead a discussion of the general formula for the sum of a geometric series. Alert students to the fact that the notation t_1 is here replaced by the constant symbol a, which emphasizes the fact that a geometric series is completely determined by three constants: a, r, and n. As a result, the general formula for a geometric series of n terms with 1st term a and common ratio r can be represented by the following expression:

$$\sum_{k=1}^{n} ar^{k-1}$$

Pay special attention to the case where $r = 1$. In this case the formula fails because it requires division by $1 - r$, that is, division by 0. Encourage students to realize that when $r = 1$, the geometric series is also an arithmetic series with 1st term 1 and common difference 0. Its sum is represented by the following equation:

$$\sum_{k=1}^{n} a = \frac{n(a+a)}{2} = na$$

Ask students to discover any other examples of a series (or sequence) that is both arithmetic and geometric. In fact, the only other such example occurs when $r = 0$, a case that presents the thorny problem of how to define the expression 0^0. Mathematicians deal with this problem by defining, for series purposes only, $0^0 = 1$. You might wish to sidestep the entire issue by assuming that $r \neq 0$, but a better approach might be to invite students to contemplate the situation in their journals. In doing so, you invite students into the everyday paradoxes that mathematicians confront.

Once students understand from the General Geometric Series transparency the formula

$$\sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^n}{1-r},\ r \neq 0,\ 1,$$

distribute copies of the Geometric Sequences and Series Graphic Organizer worksheet (p. I-4). Give students time to fill in the information they need, then allow them to pair up to check each other's work.

To wrap up the day's discussion, distribute the Geometric Sequences and Series Practice homework (p. I-6), and have students continue to work in pairs on Problem 1. Offer suggestions about ways to solve the problem, but do not give its answer. Students should rely on their understanding. They should complete the remaining problems as homework.

Days 9–11

Students work in groups to review geometric sequences and series. On Day 11 they demonstrate their understanding on the unit assessment.

Materials & Resources

- ❑ Sequences, Series, and Salaries (pp. J-2–J-3)
- ❑ Sequences, Series, and Salaries Key (p. J-4)
- ❑ Sequences, Series, and Patterns (pp. J-5–J-6)
- ❑ Sequences, Series and Patterns Key (p. J-7)
- ❑ Reflecting on Patterns (p. J-8)
- ❑ Poster board*
- ❑ Markers*
- ❑ Sequences and Series Review (pp. J-9–J-10).
- ❑ Sequences and Series Review Key (p. J-11).
- ❑ Sequences and Series test (pp. J-12–J-13)
- ❑ Sequences and Series Key (p. J-14)

* Materials or resources not included in published unit

To warm up, share the following scenario, inspired by a *FoxTrot®* comic strip (Amend, 2006, September 10):

> A math teacher offers to assign 1 second of homework for the 1st week of school, 2 seconds the 2nd week, 4 seconds the 3rd, and so on. If the amount of homework doubles every week, is this something you would agree to for the duration of a 36-week school year?

Have students work in pairs to determine whether they think the homework plan is reasonable. Encourage them to use mathematics to justify their responses. When they have finished, ask them to share their answers. By using sums of series, they should realize that after only 20 weeks there would not be enough hours in the week to keep up with the workload.

For the remainder of Day 9, students will work in groups of four on real-world applications of sequences and series. The Sequences, Series, and Salaries worksheet (pp. J-2–J-3) compares arithmetic and geometric growth in salaries. The Sequences, Series, and Patterns worksheet (pp. J-5–J-6) connects geometric sequences and series to fractal geometry—a hot topic in contemporary mathematics—through the construction of a snowflake curve. Students should work on the worksheets for the rest of the class period. Use the time to informally assess students' understanding of sequences and series. Ask individuals and groups clarifying questions that require them to synthesize their understanding. To keep students engaged, periodically instruct a student from each group to swap place with a student in another group. This gives students a chance to move around and share the thinking of different groups. (It can also curb classroom management problems that develop as students become familiar with each other.)

For homework, ask students to write in their journals at least 1 paragraph in response to each of the following questions:

- What concept(s) from this unit have been easiest to understand?
- What concept(s) have been most difficult to understand?
- What concept(s) are most interesting?
- What concept(s) are least interesting?

As students enter the classroom on Day 10, they should turn in their journal assignments. Ask them to warm up by working individually on the Reflecting on Patterns worksheet (p. J-8), a worksheet that not only focuses students' review of the material but also directly addresses Essential Question 3: "How are patterns useful in making predictions?" While students are working, quickly scan their responses to the question, "What concept(s) from this unit have been most difficult to understand?" Use this information to help focus the day's review for the unit assessment. When their individual work is complete, ask students to rejoin their groups from the previous day. Distribute a poster-size version of the worksheet to each group and ask them to synthesize their information into a single poster that represents the group's understanding. Some photocopiers can reproduce worksheets on poster paper; if such technology is not available, provide poster board and markers and allow students to recreate the worksheet themselves. Students should hang completed posters in a prominent place in the room; refer to them during the class period to help guide the rest of the day's review.

Shuffle student groups as you distribute the Sequences and Series Review worksheet (pp. J-9–J-10). As they work, circulate through the room and assign specific problems to each group for presentation at the blackboard or overhead projector. As groups present their solutions, intervene only when necessary. Guide the discussion to ensure that all questions—whether those written on their posters or asked in class—have been clarified. At the end of class, remind students to use their review sheets, their glossaries, and their graphic organizers to study for the next day's exam.

Day 11 completes the unit with the Sequences and Series Test (pp. J-12–J-13), which assesses the concepts covered in the unit.

ENHANCING STUDENT LEARNING

Selected Course Standards

H.2. Sequences and Series

d. Use sequences and series to solve real-world problems

Unit Extension

Suggested Teaching Strategies/Procedures

Materials & Resources

- ❑ Famous Fibonacci (p. K-2)
- ❑ Famous Fibonacci Key (p. K-3)

- ■ Famous Fibonacci (p. K-2) is an activity that investigates recursive sequences, Fibonacci numbers, and the Golden Ratio.
- ■ Paper-folding problems provide another interesting example of the application of geometric series. Students with Internet access might conduct a WebQuest on the work of Britney Gallivan who, as a junior in high school in 2002, solved the infamous challenge of folding a piece of paper more than 8 times. She set her record by making 9, 10, 11, and 12 folds, deriving a formula for her work (Historical Society of Pomona Valley, n.d.).

Reteaching

Suggested Teaching Strategies/Procedures

Materials & Resources

- ❑ Real-World Patterns (p. K-4)
- ❑ Real-World Patterns Key (p. K-5)
- ❑ Real-Life Applications of Sequences and Series (p. K-6)
- ❑ Real-Life Applications of Sequences and Series Key (p. K-7)

Real-World Patterns (p. K-4) and Real-Life Applications of Sequences and Series (p. K-6) provide additional opportunities for students to make the connections between sequences and series and the world around them.

Reflecting on Classroom Practice

- ■ How did collaborative groups help students progress in their understanding of the concepts?
- ■ Can students make clear distinctions between sequences and series and between arithmetic and geometric sequences?

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Contents

Unit Assignments and Assessments

Name:____________________________ Period: _____ Unit 1: The Purpose and Predictability of Patterns

Directions: Prior to starting the unit, complete the log on the next page according to the example below and distribute it to students as an organizational tool.

Day Assigned	Assignment/Assessment	In Class	Home-work	Date Due	Feedback (Completed/ Points)
Day 1	Prime Factors	X			
	Number Patterns		X		
Day 2	Station Problems	X			
	Algebra Skills		X		
Day 3	Simon Says	X			
	Simon Says 2		X		
Day 4	Movie Time	X			
	Even and Odd	X	X		
Day 5	Arithmetic Sequences and Series Practice	X	X		
Day 6	Growing Geometrically	X			
	Shrinking Geometrically	X			
	Sigma Notation	X	X		
Day 7	Summing Geometrically	X	X		
Day 8	Geometric Sequences and Series Practice	X	X		
Days 9–11	Sequences, Series, and Salaries	X			
	Sequences, Series, and Patterns	X			
	Sequences and Series Review	X			
	Sequences and Series Test	X			

Unit Assignments and Assessments

Name:____________________________ Period: _____ Unit 1: The Purpose and Predictability of Patterns

Day Assigned	Assignment/Assessment	In Class	Home-work	Date Due	Feedback (Completed /Points)

Contents

Mathematics Autobiography

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Because I want to help you achieve your best, it is important that I get to know you. Use the questions below to help you reflect on your previous experiences in math classes. On the back of this worksheet, write an organized paragraph that addresses at least 4 of these questions.

- What are your expectations for this course?
- What are your responsibilities as a student in this course?
- Why do you perform well or poorly in math classes?
- Do you prefer to work alone or in a group?
- What do you do when you get "stuck" on a problem?
- When and how do you use your calculator?
- What are your study habits? (Do you take notes? Are you organized? Do you procrastinate? Do you read the textbook?)
- How can I help you to be successful in Algebra II?
- What are the qualities of a good math teacher?

Prime Factors

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Determine the prime factorization of each of the following positive integers. Show all your work.

1. 90

2. 127

3. 324

4. 354

5. 735

6. 1024

Prime Factors Key

1. $90 = 2 \cdot 3^2 \cdot 5$
2. 127
3. $324 = 2^2 \cdot 3^4$
4. $2 \cdot 3 \cdot 59$
5. $735 = 3 \cdot 5 \cdot 7^2$
6. $1{,}024 = 2^{10}$

Introduction to Patterns in Numbers

Harvey Heinz

A bookbinder by profession, Harvey Heinz has pursued recreational mathematics as a lifelong hobby and maintains several widely acclaimed websites devoted to geometric and numeric patterns. Adapted with permission from Harvey Heinz, *Number Patterns Website*. ©2002 by Harvey Heinz.

The history of mathematics is a history of people fascinated by numbers. A driving force in mathematical development has always been the need to solve practical problems. However, humanity's innate curiosity and love of pattern has probably played an equal part in this development.

Most written records of early mathematics that have survived to modern times were actually lists of mathematical problems, i.e., recreational mathematics. Examples include the Rhind Papyrus (circa 1700 B.C.), a series of 87 problems, which was the key to deciphering Egyptian hieroglyphs; and Diophantus's *Arithmetica* (circa 250 A.D.), a collection of 130 mathematical problems with numerical solutions of determinate equations.

The Pythagoreans (circa sixth century B.C.) were a secret society who considered numbers sacred and tried to find relations between numbers and nature. For example, they developed the musical scales as number ratios. They also discovered that 6 and 28 are the first two *perfect numbers* (a perfect number is equal to the sum of its proper divisors, i.e., the divisors that differ from the number itself). The Pythagoreans also knew that squared numbers are the sums of sequences of consecutive odd numbers.

A recent example of mathematics being advanced with no apparent practical purpose was the proving of Fermat's Last Theorem, which states that when n is a positive integer greater than 2, the equation $A^n + B^n = C^n$ has $A = 0$, $B = 0$, $C = 0$ as its only solution in the integers.

If $n = 2$, there are many positive integers *A, B,* and *C* that solve this equation: these are the Pythagorean triples. But when $n \geq 3$ the equation becomes much more difficult to solve. This deceptively simple theorem, first casually written in the margins of Pierre de Fermat's copy of Diophantus's *Arithmetica* in 1637, is simple to state but has preoccupied countless mathematicians for over 350 years. It was finally proved by Professor Andrew Wiles in 1994, after nearly 8 years of work.

Number Patterns

Name: ______________________________ Period: ________________________ Date: ________________

Directions: After reading “Introduction to Patterns in Numbers” by Harvey Heinz, respond to the prompts below.

1. Use the definition of *perfect numbers* to show that 6 and 28 are perfect numbers.

2. Explain the relationship between Pythagorean triples and right triangles.

3. List 3 sets of Pythagorean triples. Show your work.

4. According to Harvey Heinz, “squared numbers are the sums of sequences of consecutive odd numbers.” Check this statement by showing that the first 5 perfect squares are the result of adding one or more consecutive odd numbers.

5. Heinz suggests that human beings have been fascinated by number patterns for thousands of years. List 4 number patterns *not mentioned in the article* that interest you.

Number Patterns Key

1. $6 = 1 + 2 + 3$
   $28 = 1 + 2 + 4 + 7 + 14$

2. Positive integers *A, B,* and *C* satisfy $A^2 + B^2 = C^2$ if and only if *A* and *B* are sides of a right triangle and *C* is the hypotenuse.

3. (3, 4, 5): $3^2 + 4^2 = 9 + 16 = 25 = 5^2$
   (5, 12, 13): $5^2 + 12^2 = 25 + 144 = 169 = 13^2$
   (7, 24, 25): $7^2 + 24^2 = 49 + 576 = 625 = 25^2$
   Any multiple of these will work as well, as can be seen by direct calculation or similar triangles.

4. $1^2 = 1$
   $2^2 = 4 = 1 + 3$
   $3^2 = 9 = 1 + 3 + 5$
   $4^2 = 16 = 1 + 3 + 5 + 7$
   $5^2 = 25 = 1 + 3 + 5 + 7 + 9$

5. Answers will vary but may include entries in Pascal's triangle, Fibonacci numbers, prime numbers, squares, or cubes.

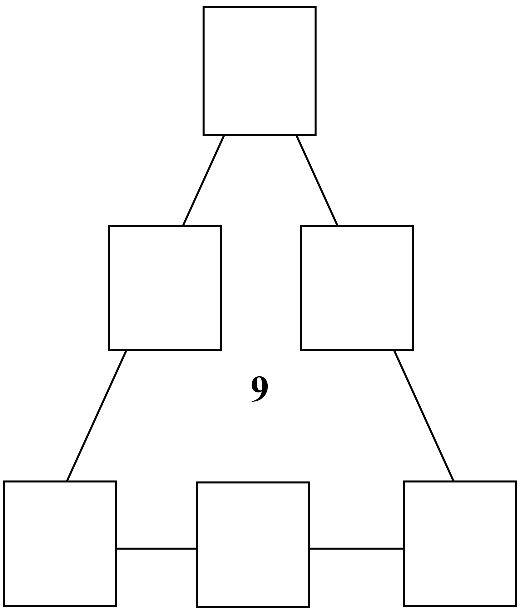
Contents

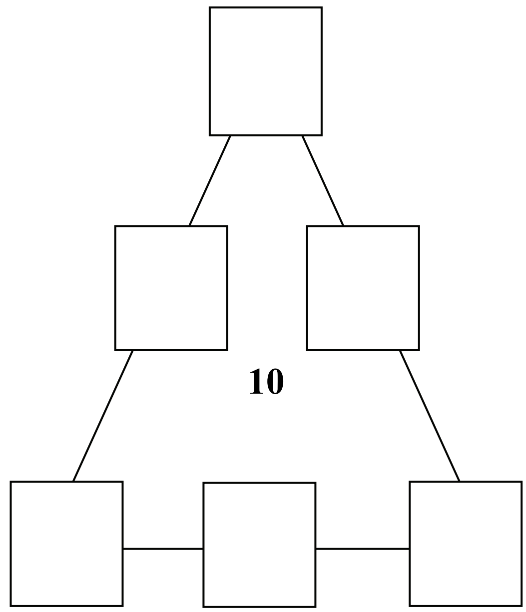
Station Problems

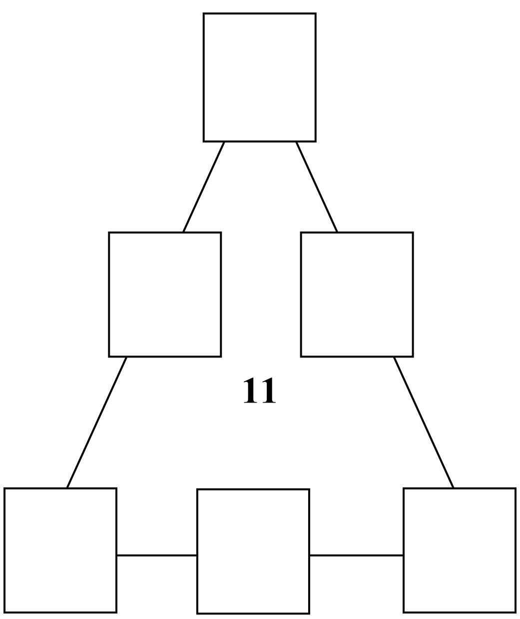
Name: ______________________________ Period: ________________________ Date: ________________

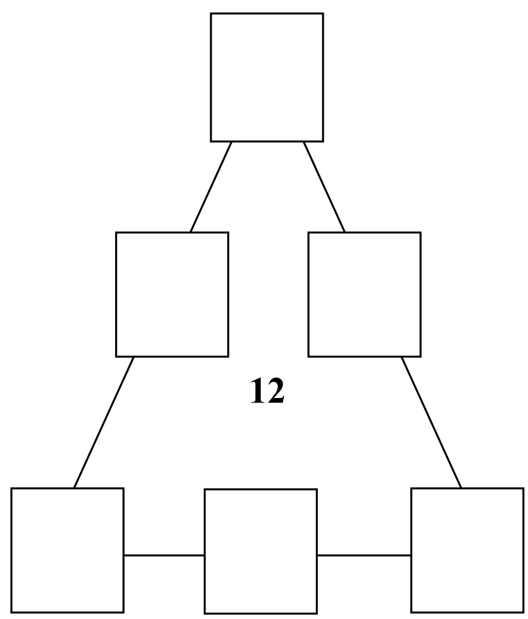
Station 1

Directions: Use the consecutive integers 1–6 to fill in the boxes in each set below. The sum of each side of the triangle should be equal to the number in the interior of the triangle. One problem per student.









Name: ______________________________ Period: ________________________ Date: ________________

Station 2

Directions: Insert any of the mathematical operation signs (+, –, ×, ÷), the equals sign (=), and grouping symbols (parentheses) into the sequence of numbers below to create two true mathematical statements. Here is an example using the integers 1 through 5: (1 + 2) × 3 = 4 + 5.

1 2 3 4 5 6 7 8

1 2 3 4 5 6 7 8

Name: ______________________________ Period: ________________________ Date: ________________

Station 3

Directions: Solve the problem and explain your strategy for solving it.

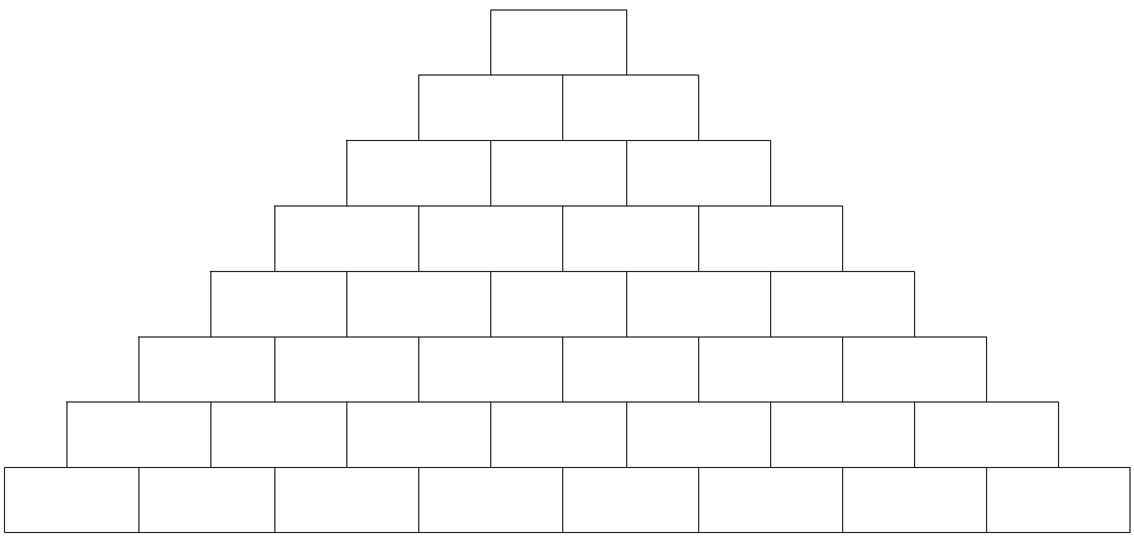
There are 25 students in your homeroom. On the last day of the school year, every student exchanges a wallet-size photograph with each of the other students. How many exchanges are necessary for each student to have exactly 1 photo of every other classmate?

Station 4

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Fill in the values of the cells in the triangle according to the explanation below. List all the patterns you see in the triangle horizontally and diagonally.

Although the Chinese first discovered it, the arithmetical arrangement of numbers known as Pascal's Triangle is named after French mathematician and scientist Blaise Pascal because he determined many of its important uses. To construct the triangle, assign the top row the value 1. We'll call this Row 0. To find the value of each remaining cell, add the values of the cells above and to the left and above and to the right and record the sum. For example, to find the value of the cell on the left in Row 1, because there is no cell above and to the left, and because the value of the cell above and to the right is 1, the sum is 1 (i.e., 0 + 1 = 1). To find the value of the cell on the right, because the value of the cell above and to the left is 1 and there is no cell above and to the right, the sum is also 1 (i.e., 1 + 0 = 1).

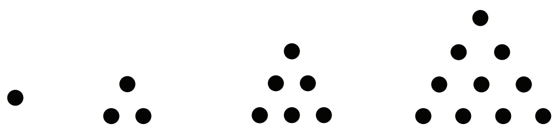


Name: ______________________________ Period: ________________________ Date: ________________

Station 5

Directions: Read the explanation and respond to the prompts below.

Polygonal numbers are numbers that are geometric representations of values created by equally spaced points. For example, 1, 3, 6, and 10 are triangular numbers that can be geometrically represented by an array of dots that forms a triangular pattern.



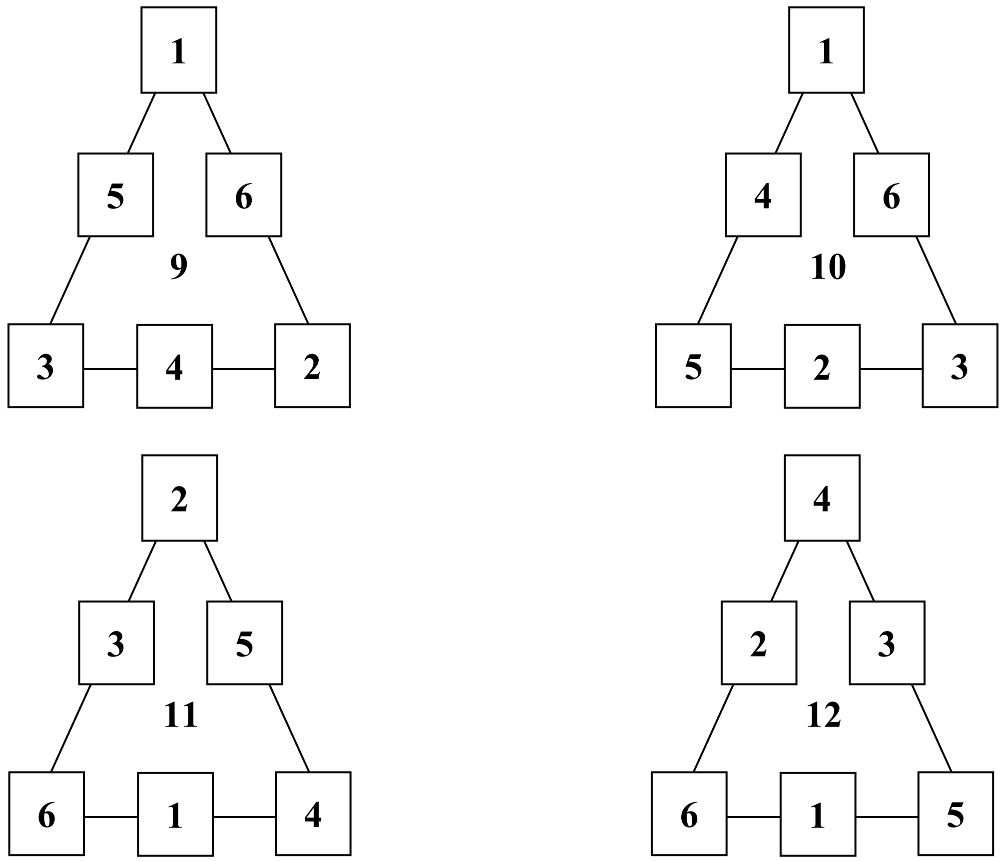
1 3 6 10

1. What are the next three triangular numbers? Draw diagrams in the space above to support your answers.

2. In the space below, create the first 5 square numbers. (Hint: 1 is the first square number.) Include diagrams to support your responses.

Station Problems Key

Station 1



Station 2

Possible answers include:

- 1 + 2 + 3 – 4 + 5 = 6 – 7 + 8
- (1 + 2 + 3 + 4) · 5 + 6 = 7 · 8

Station 3

Student 1 exchanges with every other student in class resulting in 24 picture exchanges. Student 2 exchanges with every other student (except Student 1) which results in 23 exchanges. Student 3 exchanges with every other student (except Students 1 and 2) which results in 22 exchanges. This pattern continues until there are no exchanges left. The result is:

24 + 23 + 22 + . . . + 1 = 300

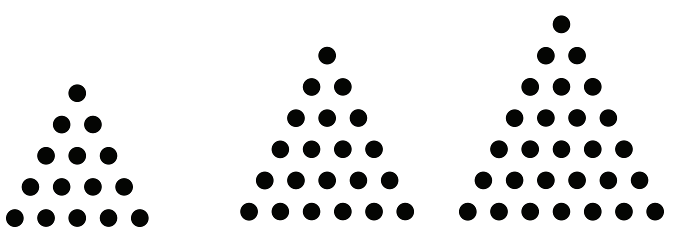
Station 4

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

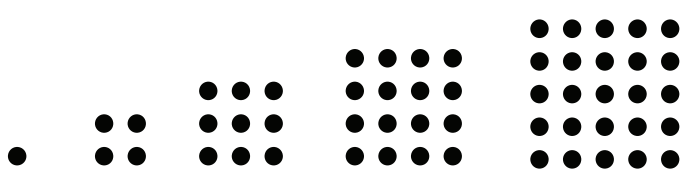
Answers will vary but could include:

- Counting numbers in one diagonal
- Triangular numbers in another diagonal
- Tetrahedral numbers in yet another diagonal
- Symmetry in a horizontal row
- Sum terms in row $r = 2^r$
- Term r in row n given by the formula ${}_nC_r$

Station 5



1. 15 21 28



2. 1 4 9 16 25

Eratosthenes' Prime Sieve

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Produce the first several terms in the sequence of prime numbers by trying this algorithm, which was first developed by the Greek mathematician Eratosthenes.

1. Circle the first number, 2.
2. Move across the rows and down the columns of the table and cross out any number greater than 2 that is divisible by 2; these numbers are all *composite* (i.e., non-prime).
3. Repeat the procedure with the next uncrossed number, 3.
4. Repeat the procedure with the next uncrossed number, keeping a list of every prime that you use, until every number has either been circled or crossed out.

The crossed-out numbers are all the composite numbers, and the circled numbers are the primes less than or equal to 100. How many prime numbers did you need to consider before every composite number in the table was crossed out?

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Eratosthenes' Prime Sieve Key

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Answer: We only had to consider the prime numbers 2, 3, 5, and 7.

Group Participation and Collaboration Rubric

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Rate your group's performance by marking an X along the continuum. Use the descriptions below each continuum as guides.

Group Participation and Collaboration

Interaction of Group

|---|---|

■ Little interaction, one person dominating	■ Some interaction, a few people contributing ideas	■ Enthusiastic interaction, everyone contributing ideas

Focus On Topic

|---|---|

■ Conversations not always on topic	■ Conversations usually focused on topic	■ Involved conversations on topic

Reflective Thinking

|---|---|

■ Few ideas contributed that encourage reflective thinking	■ Some ideas contributed that encourage reflective thinking	■ Many ideas contributed that encourage reflective thinking

Social Interaction

|---|---|

■ Interactions show little evidence of turn taking or respect for others	■ Interactions show evidence of turn taking or respect for others	■ Interactions show strong evidence of turn taking and respect for others

On-Task Behavior

|---|---|

■ Few students on task	■ Most students on task	■ All students on task

Comments

Algebra Skills

Name: ____________________________ Period: ______________________ Date: _______________

Directions: Show all your work in the space provided. Draw a circle or box around your final answer.

1. Solve each equation or inequality.

   a. $-17 = 2a + 9$

   b. $5c + 3(c - 2) = 18$

   c. $-2y < 14$

   d. $3x + 16 \geq 9$

2. Evaluate each expression if $x = 3$, $y = -2$, and $z = 5$.

   a. $y(x - z) + x^2$

   b. $xz^2 + |y|$

   c. $xz + y^2$

3. If $f(x) = 3x^2 - x - 2$, what is $f(-2)$?

4. Simplify each of the following expressions:

   a. $9 - [4 - (3 - 5)]$

   b. $2(3c^3d)^2$

5. Find the number.

a. What is 60% of 35 ? b. 4.2 is 5% of what number?

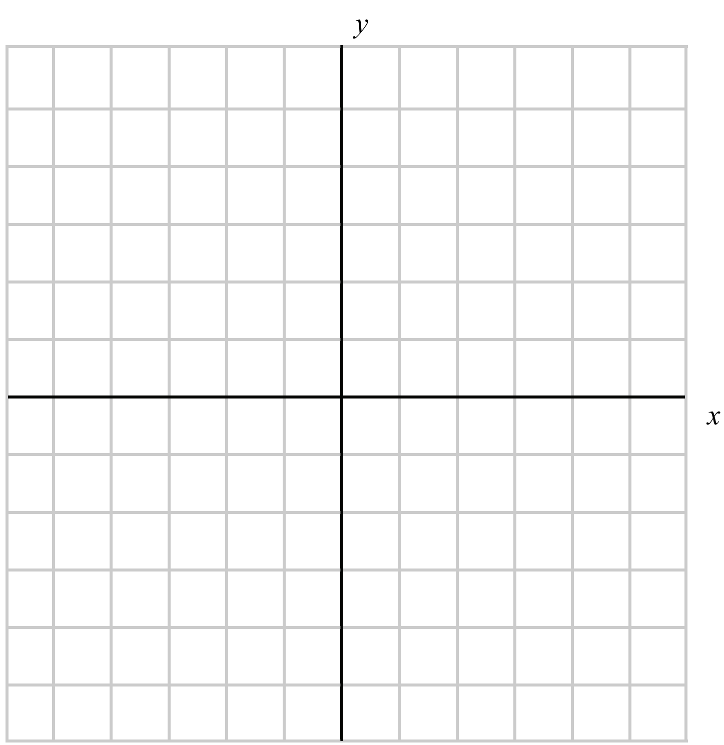
6. In what quadrant is the point

a. (–4,–1) ? b. (4,–1) ?

7. Given the equation $2y - 3 = 4x$

a. Find the slope. b. Find the y-intercept.

c. Graph the equation:



Algebra Skills Key

1. a. $a = -13$
   b. $c = 3$
   c. $y > -7$
   d. $x \geq -\frac{7}{3}$

2. a. 13
   b. 77
   c. 19

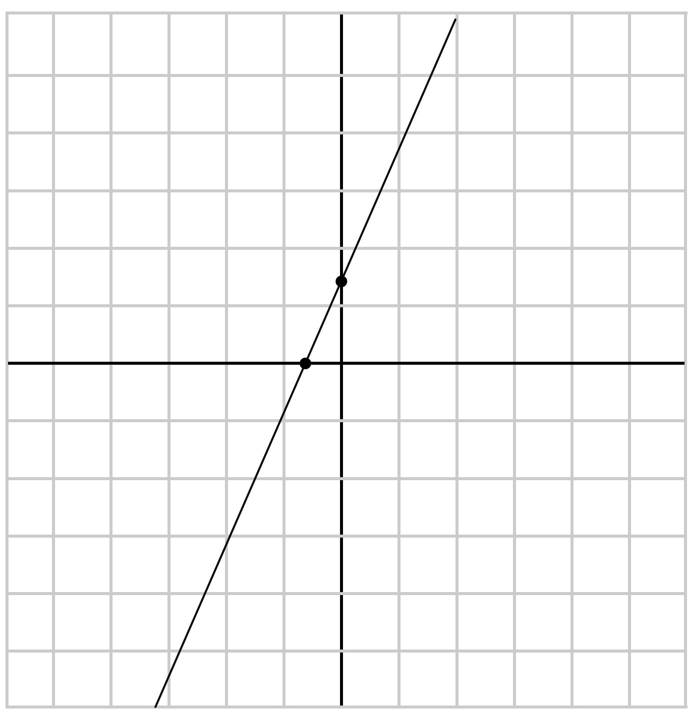
3. 12

4. a. 3
   b. $18c^6d^2$

5. a. 21
   b. 84

6. a. III
   b. IV

7. a. 2
   b. $\frac{3}{2}$
   c. The graph is a line passing through the points $(-\frac{3}{4})$. and $(0, \frac{3}{2})$.



Contents

Simon Says

Name: ______________________________ Period: ________________________ Date: ________________

Betty, Michael, and Thomas are playing Simon Says. Thomas, who is playing Simon, tells Michael and Betty to walk home, a distance of 1 mi, in segments. Michael is to walk 900 ft in each of his turns (segments). Betty is to walk $\frac{1}{2}$ of the remaining distance in each of her turns (segments). Complete a table for each person.
(Note: 1 mi = 5,280 ft)

Michael

Segment	Distance Walked	Total Distance
1		
2		
3		
4		
5		
6		
7		
8		
9		

Betty

Segment	Distance Walked	Total Distance
1		
2		
3		
4		
5		
6		
7		
8		
9		

Describe each pattern in words. Who is ahead at the end of each segment?

Simon Says Key

Michael

Michael walks a fixed distance of 900 ft in each segment. His total distance increases by a fixed increment of 900 ft per segment. The total distance Michael walks after n segments, in feet, is $900n$.

Segment	Distance Walked	Total Distance (ft)
1	900	900
2	900	1,800
3	900	2,700
4	900	3,600
5	900	4,500
6	900	5,400
7		
8		
9		

Betty

In each segment, Betty walks $\frac{1}{2}$ the distance she walked in the previous segment. Indeed, in segment n she walks a distance, in feet, of $5{,}280 \times (\frac{1}{2})^n$, though students should not be expected to know this yet. Betty is much faster than Michael for the first 2 segments, and despite her decreasing speed she remains ahead of him through the first 5 segments. Nevertheless, Michael passes her and reaches home during the 6th segment. Technically Betty never gets there.

Segment	Distance Walked	Total Distance (ft)
1	2,640	2,640
2	1320	3,960
3	660	4,620
4	330	4,950
5	165	5,115
6	82.5	5,197.5
7	41.25	5,238.75
8	20.625	5,259.375
9	10.3125	5,269.6875

Glossary

Directions: Write the vocabulary word in the 1st column. In the next 2 columns provide the definition and a sketch (if appropriate) for each word. If the word can be represented symbolically, include that in the 4th column. In the final column, list additional information to help you remember the word's meaning (e.g., examples or nonexamples).

Word	Definition	Sketch	Symbol	Notes

Class Journal Feedback Rubric

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Use this rubric to give feedback to students on their journal entries.

Effort

- ❑ **Completion:** Your journal includes all assigned work.
- ❑ **Legibility:** Your journal is readable, presentable, and coherent.
- ❑ **Use:** Your journal is used to think, learn, practice, and understand.
- ❑ **Improvement:** Your journal shows overall improvements since last time.

Writing

- ❑ **Fluency:** You write with ease about a range of subjects.
- ❑ **Development:** Your writing includes examples, details, or quotations when appropriate.

Understanding

- ❑ **Thoroughness:** Your entries show you are trying to fully understand or communicate an idea through writing or illustrations.
- ❑ **Insight:** Your writing demonstrates deep understanding of ideas and goes beyond the obvious.

Requirements

- ❑ **Format:** All entries clearly list in the margin the date and title of the entry.
- ❑ **Organization:** Entries appear in chronological sequence or as otherwise assigned.

Notes

Patterns 1

Directions: Describe each pattern.

1. –3, 2, 7, 12, . . .

2. 18, 12, 6, 0, . . .

3. $\frac{1}{2}$, 1, 2, 4, . . .

4. –192, 48, –12, 3, . . .

Patterns 1 Key

1. –3, 2, 7, 12, . . .
   Produce the next number in the list by adding 5.
2. 18, 12, 6, 0, . . .
   Produce the next number in the list by subtracting 6, i.e., by adding –6.
3. $\frac{1}{2}$, 1, 2, 4, . . .
   Produce the next number in the list by multiplying by 2.
4. –192, 48, –12, 3, . . .
   Produce the next number in the list by dividing by –4 or multiplying by $-\frac{1}{4}$.

Patterns 2

1. Each of these patterns is a _______________: a list of values that change according to some pattern. Use this pattern to answer the questions that follow: –3, 2, 7, 12, . . .

   a. Each number in the sequence is called a ____________.

   b. What is the first term? ____________

   c. The first term of a sequence is denoted by t_1 and is read "____________ _______ ____________."

   d. Each term is named consecutively after t_1. What is t_3 in the sequence above? ____________

   e. In moving from one term to the next, a constant amount _____________ is added.

   f. Looking at it another way, if you take the difference between any of the terms and the one that came before it, you always obtain the same value, _____________, which is called the ____________ ____________, and denoted by d.

   g. Because each term is related to the one that precedes it by subtracting the common difference, the sequence is called an ____________ ____________.

2. Now use this next pattern to answer the following questions: 18, 12, 6, 0, . . .

   a. d = ____________

   b. t_1 = ____________

   c. If the pattern continues, what will be the value of t_7? ____________

Patterns 2 Key

1. *sequence*
   a. *term*
   b. –3
   c. *tee sub one*
   d. 7
   e. 5
   f. 5, *common difference*
   g. *arithmetic sequence*
2. a. –6
   b. 18
   c. –18

Simon Says 2

Name: ______________________________ Period: ________________________ Date: ________________

Part I

Directions: Answer each of the following questions. Be sure to explain your answers.

In the Simon Says class activity, Michael made it home by walking 900 ft in each segment of his trip. Let's look at the first 5 segments Michael walked:

Segment	Distance Walked (ft)	Total Distance (ft)
1	900	900
2	900	1,800
3	900	2,700
4	900	3,600
5	900	4,500

1. Look at the 2nd column of the table, which expresses the distance Michael walked per segment, in feet.

   a. Express the distance walked per segment as a sequence of 5 terms:

      t_1 = ________
      t_2 = ________
      t_3 = ________
      t_4 = ________
      t_5 = ________

   b. Is the sequence an arithmetic sequence? Explain why or why not. If your answer is yes, what is the value of the common difference, d ?

2. Look at the 3rd column of the table, which expresses the total distance Michael walked, in feet, by the end of a given segment.

   a. Express the total distance walked as a sequence of 5 terms:

      t_1 = ________
      t_2 = ________
      t_3 = ________
      t_4 = ________
      t_5 = ________

   b. Is the sequence an arithmetic sequence? Explain why or why not. If your answer is yes, what is the value of the common difference, d ?

Part 2

Directions: Answer each of the following questions. Be sure to explain your answers.

In the Simon Says class activity, Betty set out toward home a mile away by walking $\frac{1}{2}$ the remaining distance in each segment. As a result, she never quite made it home. Let's look at the first 5 segments Betty walked:

Segment	Distance Walked (ft)	Total Distance (ft)
1	2,640	2,640
2	1,320	3,960
3	660	4,620
4	330	4,950
5	165	5,115

3. Look at the 2nd column of the table, which expresses the distance Betty walked per segment, in feet.

   a. Express the distance walked per segment as a sequence of 5 terms:
   t_1 = ________
   t_2 = ________
   t_3 = ________
   t_4 = ________
   t_5 = ________

   b. Is the sequence an arithmetic sequence? Explain why or why not. If your answer is yes, what is the value of the common difference, d ?

4. Look at the 3rd column of the table, which expresses the total distance Betty walked, in feet, by the end of a given segment.

   a. Express the total distance walked as a sequence of 5 terms:
   t_1 = ________
   t_2 = ________
   t_3 = ________
   t_4 = ________
   t_5 = ________

   b. Is the sequence an arithmetic sequence? Explain why or why not. If your answer is yes, what is the value of the common difference, d ?

Simon Says 2 Key

Part 1

1. a. $t_1 = 900$
   $t_2 = 900$
   $t_3 = 900$
   $t_4 = 900$
   $t_5 = 900$

   b. The sequence is arithmetic with a common difference $d = 0$.

2. a. $t_1 = 900$
   $t_2 = 1{,}800$
   $t_3 = 2{,}700$
   $t_4 = 3{,}600$
   $t_5 = 4{,}500$

   b. The sequence is arithmetic with a common difference $d = 900$.

Part 2

3. a. $t_1 = 2{,}640$
   $t_2 = 1{,}320$
   $t_3 = 660$
   $t_4 = 330$
   $t_5 = 165$

   b. This is not an arithmetic sequence because there is no common difference between terms:
   $t_2 - t_1 = -1{,}320$, $t_3 - t_2 = -660$, $t_4 - t_3 = -330$, $t_5 - t_4 = -165$
   (These differences are precisely the opposites of the last 4 terms of the sequence.)

4. a. $t_1 = 2{,}640$
   $t_2 = 3{,}960$
   $t_3 = 4{,}620$
   $t_4 = 4{,}950$
   $t_5 = 5{,}115$

   b. This is not an arithmetic sequence because there is no common difference between terms:
   $t_2 - t_1 = 1{,}320$, $t_3 - t_2 = 660$, $t_4 - t_3 = 330$, $t_5 - t_4 = 165$

Contents

Three Sequences

Directions: Find the indicated values for each sequence.

1. –1, 3, 7, . . .
   a. $t_1 =$
   b. $d =$
   c. $t_7 =$

2. 5, 2, –1, . . .
   a. $t_1 =$
   b. $d =$
   c. $t_7 =$

3. 22.3, 17.6, 12.9, . . .
   a. $t_1 =$
   b. $d =$
   c. $t_7 =$

Three Sequences Key

1. a. $t_1 = -1$
   b. $d = 4$
   c. $t_7 = 23$

2. a. $t_1 = 5$
   b. $d = -3$
   c. $t_7 = -13$

3. a. $t_1 = 22.3$
   b. $d = -4.7$
   c. $t_7 = -5.9$

Movie Time

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Read the prompt and respond to the questions. Be sure to explain how you found your answer.

The after-prom party committee at your high school has rented the local movie theater. As you walk into the theater, you notice that:

- The 1st row has 15 seats.
- The 2nd row has 17 seats.
- Each successive row has 2 additional seats.
- There are 20 rows total in the theater.

A total of 723 students will attend the party. You need to know if there are enough seats for all of them in the theater.

1. How many seats are in the 3rd row?

2. How many seats are in the 7th row?

3. What is the total number of seats in Rows 1–4 ?

4. What is the total number of seats in the theater? Explain your strategy for determining your answer.

5. Are there enough seats in the theater to accommodate all students attending the prom? If not, how many students will be without a seat in the theater?

Movie Time Key

1. There are 19 seats in Row 3.
2. There are 27 seats in Row 7.
3. There are 72 seats in Rows 1–4.
4. Pairing Rows 1 and 20, 2 and 19, 3 and 18, an so forth, yields 10 pairs, each containing 68 seats. So the total number of seats in the theater is 10 × 68 = 680. (This same kind of pairing can be used to answer Question 3: the 4 rows may be divided into 2 pairs, each containing 36 seats. The total number of seats in the first 4 rows is 2 × 36 = 72.)
5. There are not enough seats in the theater to accommodate everyone who attends the party. There are 723 – 680 = 43 students who will not have a seat in the theater.

Row	Seats
1	15
2	17
3	19
4	21
5	23
6	25
7	27
8	29
9	31
10	33
11	35
12	37
13	39
14	41
15	43
16	45
17	47
18	49
19	51
20	53

Summing It Up

Part 1

We will use the formula for the number of seats in the *n*th row (for $1 \le n \le 20$) to calculate the total number of seats in the first 4 rows of the theater.

$$t_n = 15 + 2(n - 1)$$

Let S denote the sum of the terms t_1 through t_4. The commutative property tells us that we can compute the sum in any order we wish. We'll compute the sum both forward and backward:

$$\begin{array}{rcllll} S &=& t_1 & + \; t_2 & + \; t_3 & + \; t_4 \\ S &=& t_4 & + \; t_3 & + \; t_2 & + \; t_1 \\ \hline 2S &=& (t_1 + t_4) & + \; (t_2 + t_3) & + \; (t_3 + t_2) & + \; (t_4 + t_1) \end{array}$$

Adding the sums together gives us an expression for $2S$ as the sum of 4 terms; let's take a closer look at those terms:

$$\begin{aligned} t_1 + t_4 &= [15 + 2(1 - 1)] + [15 + 2(4 - 1)] \\ &= 15 + 21 \\ &= 36 \end{aligned}$$

$$\begin{aligned} t_2 + t_3 &= [15 + 2(2 - 1)] + [15 + 2(3 - 1)] \\ &= 17 + 19 \\ &= 36 \end{aligned}$$

1. Compute the values of the remaining 2 terms in the expression for $2S$.
2. What do you observe?
3. What value do you get for $2S$?
4. What value do you get for S ?

Part 2

We will use the formula for the number of seats in the *n*th row (for $1 \le n \le 20$) once again, this time to calculate the total number of seats in the first 20 rows of the theater.

$$t_n = 15 + 2(n - 1)$$

Let S denote the sum of the terms t_1 through t_{20}. Once again, we'll compute the sum both forward and backward:

$$\begin{aligned} S &= t_1 + t_2 + t_3 + \ldots + t_{18} + t_{19} + t_{20} \\ S &= t_{20} + t_{19} + t_{18} + \ldots + t_3 + t_2 + t_1 \\ \hline 2S &= (t_1 + t_{20}) + (t_2 + t_{19}) + (t_3 + t_{18}) + \ldots + (t_{18} + t_3) + (t_{19} + t_2) + (t_{20} + t_1) \end{aligned}$$

Adding the 2 sums together gives us an expression for $2S$ as the sum of 10 terms; let's take a closer look at those terms:

$$\begin{aligned} t_1 + t_{20} &= [15 + 2(1 - 1)] + [15 + 2(20 - 1)] \\ &= (15 + 0) + (15 + 38) \\ &= 15 + 53 \\ &= 68 \end{aligned}$$

$$\begin{aligned} t_2 + t_{19} &= [15 + 2(2 - 1)] + [15 + 2(19 - 1) \\ &= (15 + 2) + (15 + 36) \\ &= 17 + 51 \\ &= 68 \end{aligned}$$

1. Compute the values of the remaining 18 terms in the expression for $2S$.
2. What do you observe?
3. What value do you get for $2S$?
4. What value do you get for S ?

Summing It Up Key

Part 1

1. The two remaining terms are identical to the terms just calculated.
2. Students may observe that $t_3 + t_2 = t_2 + t_3 = 36$ and $t_4 + t_1 = t_1 + t_4 = 36$ by the commutative property of addition.
3. $2S = 4(36) = 144$
4. $S = 2(36) = 72$

Part 2

1. Students should soon notice that each of the 20 terms in the sum is equal to 68.
2. Students who are particularly comfortable using notation together with the distributive property may notice that if k is an integer and $1 \le k \le 20$, then

$$\begin{aligned} t_k + t_{21-k} &= [15 + 2(k-1)] + \{15 + 2[(21-k)-1] \\ &= 2(15) + 2\{(k-1) + [(21-k)-1]\} \\ &= 2(15) + 2[(k-1) + (21-k-1)] \\ &= 2(15) + 2(21-2) \\ &= 2(15) + 2(19) \\ &= 2(34) = 68 \end{aligned}$$

3. $2S = 20(68) = 1360$
4. $S = 10(68) = 680$

Even and Odd

Name: ____________________________ Period: ______________________ Date: _______________

Directions: Answer each of the following questions. Be sure to explain your answers.

1. Consider the first 10 even natural numbers 2, 4, 6, 8, 10, 12, 14, 16, 18, 20 as a sequence with *n*th term a_n.
   a. Explain why the numbers are an arithmetic sequence. How many terms does the sequence have?

   b. Determine the value of the first term a_1 and the common difference d.

   c. Give a formula for the *n*th term a_n.

   d. Set S_n equal to the sum of the first n even natural numbers. Use the doubling method to calculate the values of S_5, S_7, and S_{10}. Determine a general formula for S_n.

2. Consider the first 10 odd natural numbers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 as a sequence with nth term b_n.

   a. Explain why the numbers are an arithmetic sequence. How many terms does the sequence have?

   b. Determine the value of the first term b_1 and the common difference d.

   c. Give a formula for the nth term b_n.

   d. Set S_n equal to the sum of the first n odd natural numbers. Use the doubling method to calculate the values of S_5, S_7, and S_{10}. Determine a general formula for S_n.

3. Suppose you have been given an arithmetic sequence of 10 terms with first term c_1 and common difference d. Generalizing from the formulas you have already developed, infer a formula for the nth term c_n and the sum S_{10}. Explain your formula.

Even and Odd Key

1. a. This is an arithmetic sequence because the difference between any given term after the first and the one preceding it is constantly equal to 2. It has 10 terms.

   b. $a_1 = 2$, $d = 2$

   c. $a_n = 2n$

   d. Here is an illustration of the doubling method for S_5:

S_5	=	2	+	4	+	6	+	8	+	10
S_5	=	10	+	8	+	6	+	4	+	2
$2S_5$	=	12	+	12	+	12	+	12	+	12
$2S_5$	=	5(12)	=	60						
S_5	=	30								

   Similar calculations show that $S_7 = 56$ and $S_{10} = 110$. Students may conjecture that $S_n = n(n + 1)$. They may also realize that $S_n = (\frac{n}{2})(a_1 + a_n)$.

2. a. This is an arithmetic sequence because the difference between any given term after the first and the one preceding it is constantly equal to 2. It has 10 terms.

   b. $b_1 = 1$, $d = 2$

   c. $b_n = 2n - 1$

   d. Here is an illustration of the doubling method for S_5:

S_5	=	1	+	3	+	5	+	7	+	9
S_5	=	9	+	7	+	5	+	3	+	1
$2S_5$	=	10	+	10	+	10	+	10	+	10
$2S_5$	=	5(10)	=	50						
S_5	=	25								

   Similar calculations show that $S_7 = 49$ and $S_{10} = 100$. Students may conjecture that $S_n = n^2$. They may also realize that $S_n = (\frac{n}{2})(b_1 + b_n)$.

3. By now students may realize that $c_n = c_1 + d(n - 1)$. They may also realize that $S_{10} = (\frac{10}{2})(c_1 + c_{10})$. Using the formula for c_n, they may also observe that $c_1 + c_{10} = 2c_1 + 9d$, and hence $S_{10} = (\frac{10}{2})(2c_1 + 9d)$. (This is where discussion begins on Day 5.)

Contents

Summing It Up Again

Part 1

Suppose $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}$ is an arithmetic sequence of 10 terms with common difference d. How can we find a formula for the sum S_{10} of the sequence's 10 terms?

1. First, find a formula for the nth term of the sequence, c_n. To begin, list the first 3 terms in the sequence:

   $c_1 = c_1$

   $c_2 = c_1 + d$

   $c_3 = c_2 + d$

2. Now, substitute the expression for c_2 from the 2nd equation into the 3rd equation to get:

   $$\begin{aligned} c_3 &= (c_1 + d) + d \\ &= c_1 + 2d \end{aligned}$$

3. Calculate the next element in the same way:

   $$\begin{aligned} c_4 &= c_3 + d \\ &= (c_1 + 2d) + d \\ &= c_1 + 3d \end{aligned}$$

4. Calculate the remaining terms, then surmise a general formula for the nth term of the sequence, c_n.

Part 2

Suppose $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}$ is an arithmetic sequence of 10 terms with common difference d. In Part I we found a formula for the sequence's nth term:

$$c_n = c_1 + (n - 1)d, \text{ when } 1 \leq n \leq 10$$

1. Use the formula for c_n to find a formula for the sum S_{10} of the sequence's 10 terms. Calculate S_{10} using the doubling method:

$$\begin{aligned} S_{10} &= c_1 + c_2 + c_3 + \ldots + c_8 + c_9 + c_{10} \\ S_{10} &= c_{10} + c_9 + c_8 + \ldots + c_3 + c_2 + c_1 \\ \hline 2S_{10} &= (c_1 + c_{10}) + (c_2 + c_9) + (c_3 + c_8) + \ldots + (c_8 + c_3) + (c_9 + c_2) + (c_{10} + c_1) \end{aligned}$$

2. Calculate the 1st term in the sum for $2S_{10}$:

$$\begin{aligned} c_1 + c_{10} &= c_1 + [c_1 + 10 - 1)d] \\ &= 2c_1 + 9d \end{aligned}$$

3. Calculate the next 2 terms. (Remember that $c_2 = c_1 + d$ and $c_9 = c_{10} - d$.)

$$\begin{aligned} c_2 + c_9 &= (c_1 + d) + (c_{10} - d) \\ &= c_1 + c_{10} \\ &= 2c_1 + 9d \end{aligned}$$

$$\begin{aligned} c_3 + c_8 &= (c_2 + d) + (c_9 - d) \\ &= c_2 + c_9 \\ &= c_1 + c_{10} \\ &= 2c_1 + 9d \end{aligned}$$

4. Calculate the remaining terms of the sum.
   a. What happens?
   b. What is the value of $2S_{10}$?
   c. What is the value of S_{10}?

Summing It Up Again Key

Part 1

4. $c_5 = c_1 + 4d$
   $c_6 = c_1 + 5d$
   $c_7 = c_1 + 6d$
   $c_8 = c_1 + 7d$
   $c_9 = c_1 + 8d$
   $c_{10} = c_1 + 9d$
   In general, $c_n = c_1 + (n - 1)d$ for $1 \leq n \leq 10$.

Part 2

4. a. Each of the 10 terms of the sum is equal to $c_1 + c_{10} = 2c_1 + 9d$.

   b. $2S_{10} = 10(c_1 + c_{10})$ or $2S_{10} = 10(2c_1 + 9d)$

   c. $S_{10} = 5(c_1 + c_{10})$ or $S_{10} = 5(2c_1 + 9d)$

Arithmetic Sequences and Series

Are there rules that work for any sequence, of any length whatsoever?

Sequences and Series in General

Suppose that k is a natural number and $t_1, t_2, t_3 \ldots t_k$ is a sequence of k terms. The sum $t_1 + t_2 + t_3 + \ \ldots + t_k$ is called a *series* of k terms, and the value of the sum, S_k, is called the *sum* of the series.

1. What is the difference between a *sequence* and a *series*?

Arithmetic Sequences and Series

Consider what happens when $t_1, t_2, t_3 \ldots t_k$ is an *arithmetic sequence* of k terms with common difference d. Summing It Up Again gives a general formula for the nth term of the sequence.

$$t_n = t_1 + (n - 1)d$$

The formula is good for $1 \leq n \leq k$.

Now, find a general formula for the sum S_k of the series $t_1 + t_2 + t_3 + \ \ldots \ + t_k$, using the doubling method:

$$\begin{array}{llllllll}
S_k = & t_1 & + t_2 & + t_3 & + \ \ldots & + t_{k-2} & + t_{k-1} & + t_k \\
S_k = & t_k & + t_{k-1} & + t_{k-2} & + \ \ldots & + t_3 & + t_2 & + t_1 \\
\hline
2S_k = & (t_1 + t_k) & + (t_2 + t_{k-1}) & + (t_3 + t_{k-2}) & + \ \ldots & + (t_{k-2} + t_3) & + (t_{k-1} + t_2) & + (t_k + t_1)
\end{array}$$

2. Show that each of the terms in the final sum is equal to $t_1 + t_k$.
3. What is another expression for the sum $t_1 + t_k$?
4. Can these results be used to sum the series?

Arithmetic Sequences and Series Key

1. A sequence is an ordered list of numbers; a series is the an ordered sum of numbers.

2. $t_2 + t_{k-1} = (t_1 + d) + (t_k - d) = t_1 + t_k$
   $t_3 + t_{k-2} = (t_2 + d) + (t_{k-1} - d) = t_2 + t_{k-1} = t_1 + t_k$
   Each term equals $t_1 + t_k$.

3. $t_1 + t_k = t_1 + [t_1 + (k-1)d]$
   $= 2t_1 + (k-1)d$

4. $2S_k = k(t_1 + t_k)$

   $S_k = \frac{k}{2}(t_1 + t_k)$ or

   $S_k = \frac{k}{2}[2t_1 + (k-1)d]$

Arithmetic Sequences and Series Graphic Organizer

Name: ______________________________ Period: ________________________ Date: ________________

Arithmetic Sequence

Definition

Examples

Nonexamples

Rules & Formulas

Arithmetic sequences are expressed in the form t_1, t_2, t_3, . . . t_n.

1. Each value in the sequence is called a ______________________________.
2. The difference between consecutive terms in a sequence is called the ______________________________.
3. t_n is called the _________________________. It can be found by using the formula:

   $t_n =$

Practice

Use the *n*th term formula to find the general rule for generating a term in this sequence: 1, 5, 9, 13, . . .

Arithmetic Series

Definition

Examples

Nonexamples

Rules & Formulas

Instead of adding each term in an arithmetic series, we can find the sum by using the formula:

$S_n =$

Where:

S_n represents ____________________

t_1 represents ____________________

t_n represents ____________________

n represents ____________________

Practice

Find the sum of the first 15 terms of the series 2 + 10 + 18 + 26 + . . . (*Hint*: Find the values of each variable listed above and calculate S_n.)

Arithmetic Sequences and Series Graphic Organizer Key

Arithmetic Sequence

Definition

An ordered list of numbers in which the difference between each number and the one preceding it is constant.

Examples

Answers will vary, but may include:

- sequence of odd numbers
- sequence of even numbers
- Michael's Simon Says sequences

Nonexamples

Answers will vary, but may include:

- sequence of primes
- sequence of perfect squares
- Betty's Simon Says sequences

Rules & Formulas

Arithmetic sequences are expressed in the form t_1, t_2, t_3 . . . t_n.

1. Each value in the sequence is called a term.
2. The difference between consecutive terms in a sequence is called the common difference.
3. t_n is called the *n*th term. It can be found by using the formula:

   $t_n = t_1 + (n - 1)d$

Practice

Use the *n*th term formula to find the general rule for generating a term in this sequence: 1, 5, 9, 13, . . .

$t_n = 1 + (n - 1)d$; since $d = 4$,
this becomes $t_n = 1 + 4(n - 1) = 4n - 3$

Arithmetic Series

Definition

The sum of the terms of an arithmetic sequence.

Examples

Answers will vary, but may include:

- series of odd numbers
- series of even numbers
- Michael's Simon Says series

Nonexamples

Answers will vary, but may include:

- series of primes
- series of perfect squares
- Betty's Simon Says series

Rules & Formulas

Instead of adding each term in an arithmetic series, we can find the sum by using the formula:

$$S_n = \frac{n}{2}(t_1 + t_n), \text{ or } S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$

Where:

S_n represents the sum of the first n terms
t_1 represents the 1st term of the series
t_n represents the nth term of the series
n represents the number of terms in the series

Practice

Find the sum of the first 15 terms of the series 2 + 10 + 18 + 26 + . . . (*Hint*: Find the values of each variable listed above and calculate S_n.)

$t_1 = 2$, $d = 8$,

$t_{15} = 2 + (14)(8) = 114$

$S_{15} = \frac{15}{2}(2 + 114)$
$= 15\frac{116}{2} = 15(58) = 870$

Arithmetic Sequences and Series Practice

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Respond to each of the following prompts on a separate sheet of paper. Use your Arithmetic Sequences and Series Graphic Organizer as a reference, but be sure to indicate what facts and formulas you used and explain all answers completely.

1. Given the sequence 7, 11, 15, 19, . . .
   a. Why is this an arithmetic sequence?
   b. What is the common difference d ?
   c. What is the 1st term? The 7th term? The 10th term?
   d. What is the sum of the first 7 terms of the sequence?

2. Given the sequence 3, –8, –19, –30, . . .
   a. Why is this an arithmetic sequence?
   b. What is the common difference d ?
   c. What is the 1st term? The 10th term? The 17th term?
   d. What is the sum of the first 17 terms of the sequence?

3. Given the series 2 + 8 + 14 + . . .
   a. Why is this an arithmetic series?
   b. Find the sum S_{10}.
   c. If S_n = 1,704, what is the value of n ?

4. Given the series –7 + (–4) + (–1) + . . .
   a. Why is this an arithmetic series?
   b. Find the sum S_{13}.
   c. If S_n = 210, what is the value of n ?

Arithmetic Sequences and Series Practice Key

1. a. The sequence is arithmetic because the difference between each term and the one preceding is equal to 4.

   b. $d = 4$

   c. $t_1 = 7$; $t_7 = t_1 + d(7 - 1) = 7 + 4(6) = 31$; $t_{10} = t_1 + d(10 - 1) = 7 + 4(9) = 43$

   d. $S_7 = \frac{7}{2}(t_1 + t_7) = \frac{7}{2}(7 + 31) = \frac{7(38)}{2} = 133$

2. a. The sequence is arithmetic because the difference between each term and the one preceding is equal to –11.

   b. $d = -11$

   c. $t_1 = 3$; $t_{10} = t_1 + d(10 - 1) = 3 + (-11)(9) = -96$; $t_{17} = t_1 + d(17 - 1) = 3 + (-11)(16) = -173$

   d. $S_{17} = \frac{17}{2}(3 - 173) = \frac{17}{2}(-170) = \frac{17(-170)}{2} = 17(-85) = -1{,}445$

3. a. The series is arithmetic because the difference between each term and the one preceding is equal to 6, which is the common difference *d.*

   b. $S_{10} = \frac{10}{2}(t_1 + t_{10});\ t_1 = 2,\ t_{10} = 2 + 6(9) = 56$

   $S_{10} = \frac{10}{2}(58) = 290$

   c. $1{,}704 = \frac{n}{2}[2(2) + (n-1)(6)] = 2n + 3n(n-1) = 3n^2 - n$

   $0 = 3n^2 - n - 1{,}704 = (3n + 71)(n - 24)$

   The only other natural number solution is $n = 24$.

4. a. The series is arithmetic because the difference between each term and the one preceding it is equal to 3, which is the common difference *d*.

   b. $S_{13} = 143$

   $S_{13} = \frac{13}{2}(t_1 + t_{13});\ t_1 = -7;\ t_{13} = -7 + 12(3) = 29$

   $S_{13} = \frac{13}{2}(-7 + 29) = \frac{13(22)}{2} = 143$

   c. $210 = \frac{n}{2}[2(-7) + (n - 1)(3)] = -7n + \frac{3n(n-1)}{2}$

   Multiplying by 2 and rearranging gives: $0 = 3n^2 - 17n - 420 = (3n + 28)(n - 15)$, for which the only natural number solution is $n = 15$.

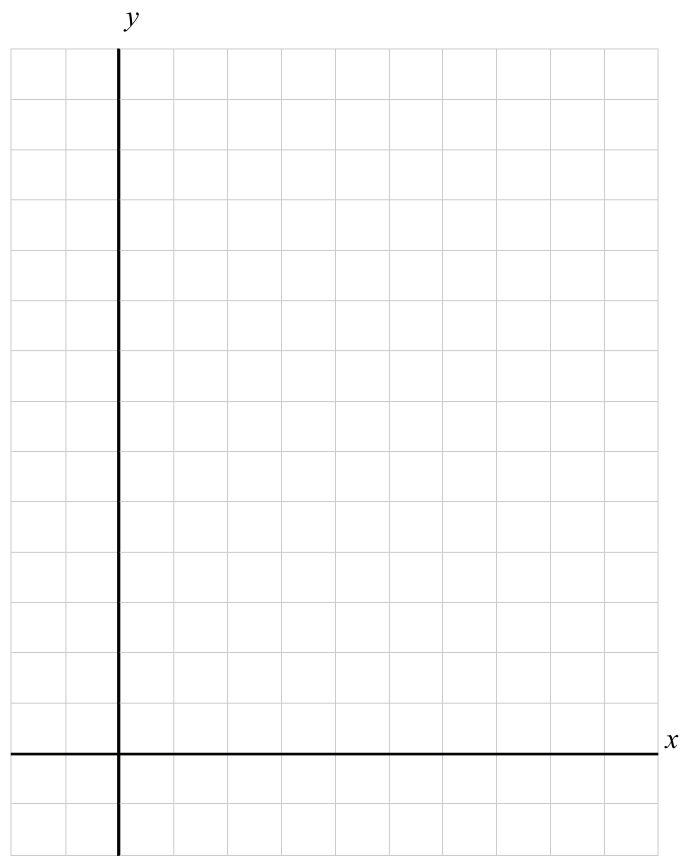
Contents

Lining Up

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Follow the directions to create a graph. Respond to each of the prompts as thoroughly as you can.

Given the sequence 21, 16, 11, 6, 1, . . .



1. Let t_n denote the nth term of the sequence. Plot the points $(1,t_1)$, $(2,t_2)$, $(3,t_3)$, $(4,t_4)$, and $(5,t_5)$ on the graph. Do not connect the points—they are discrete values!

2. What type of pattern do you observe in the graph?

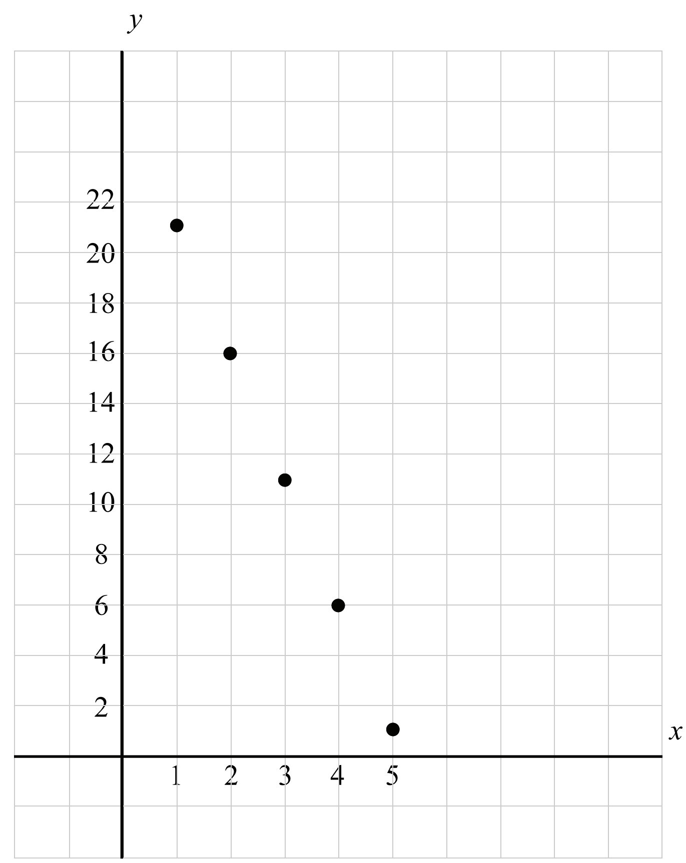
3. Write an equation for the nth term of the sequence, t_n.

4. The slope-intercept form of the equation of a line is $y = mx + b$, where m is the slope and b is the y-intercept. Explain how the parts of the nth term rule are related to this equation.

5. If you have a graphing calculator, create a scatterplot of your points. (Watch your window!) On the "Y =" screen, type the linear equation that you believe is related to your points. (If your conjecture is true, then the line should plot directly over the points.) What equation did you enter?

Lining Up Key

1. See graph at right.
2. The graph is linear.
3. Because it is a sequence with 1st term equal to 21 and common difference –5, the equation is $t_n = 21 + (-5)(n - 1)$, or $t_n = -5n + 26$
4. If n takes the place of the independent variable x and t_n takes the place of the dependent variable y, then –5 is the slope and 26 is the y-intercept.
5. $y = -5x + 26$



Growing Geometrically

Name: ____________________________ Period: ______________________ Date: ________________

Directions: Respond to each prompt below. Show your work and explain your answers.

You forwarded an e-mail chain letter to 10 friends. Let's call these 10 friends the 1st generation of recipients. Then, each of your friends forwarded the chain letter on to 10 friends, creating a 2nd generation of recipients, each of whom sent it on to 10 more friends—that is, a 3rd generation of recipients.

1. What are the first 6 terms of a sequence whose *n*th term is the number of recipients in the *n*th generation?

2. If each copy of the chain letter uses 8 kilobytes of computer storage space, what are the first 6 terms of a sequence whose *n*th term is the number of kilobytes of storage space required to store the *n*th generation of e-mail messages? (Use a different symbol for the *n*th term than the one you used in Problem 1.)

3. At the end of 6 generations, how many e-mails have been sent? How much storage space has been used?

4. If you and all the generations who receive the chain letter save all of your e-mail messages on the same Web server that has a storage capacity of 1 terabyte, in how many generations will the server be full? (Note: 1 terabyte = 1 billion kilobytes.)

Growing Geometrically Key

1. $t_1 = 10$
   $t_2 = 10t_1 = 10(10) = 10^2 = 100$
   $t_3 = 10t_2 = 10(100) = 10^3 = 1{,}000$
   $t_4 = 10t_3 = 10(1{,}000) = 10^4 = 10{,}000$
   $t_5 = 10t_4 = 10(10{,}000) = 10^5 = 100{,}000$
   $t_6 = 10t_5 = 10(100{,}000) = 10^6 = 1{,}000{,}000$

2. Define a sequence s_n—*s* for *storage space*—whose *n*th term is equal to the number of kilobytes of storage space for the *n*th generation of messages. Since each message requires 8 kilobytes of storage, s_n is equal to $8t_n$. Here are the first 6 terms of the sequence:
   $s_1 = 8t_1 = 80$
   $s_2 = 8t_2 = 800 = 8 \times 10^2$
   $s_3 = 8t_3 = 8{,}000 = 8 \times 10^3$
   $s_4 = 8t_4 = 80{,}000 = 8 \times 10^4$
   $s_5 = 8t_5 = 800{,}000 = 8 \times 10^5$
   $s_6 = 8t_6 = 8{,}000{,}000 = 8 \times 10^6$

3. Total number of e-mail messages sent: $t_1 + t_2 + t_3 + t_4 + t_5 + t_6 = 1{,}111{,}110$;
   Total amount of storage used: $s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 8{,}888{,}880$ kilobytes.

4. After *n* generations, the quantity of storage used (in kilobytes) is equal to the decimal number formed by taking *n* digits of 8 followed by 0. This number is just under $9 \times 10^{n+1}$. The server fills up during the generation when this number exceeds 10^9, i.e., during the 9th generation. (As it is, the server is nearly full at the end of the 8th generation.)

Shrinking Geometrically

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Complete the worksheet. Show all your work and explain your answers.

On the Simon Says worksheet, Betty set out toward her home a mile (5,280 ft) away by traveling $\frac{1}{2}$ the remaining distance in each segment of her walk. As a result, she never quite made it home. Here is a table that represents her trip.

Segment	Distance Walked (ft)	Total Distance (ft)
1	2,640	2,640
2	1,320	3,960
3	660	4,620
4	330	4,950
5	165	5,115
6	82.5	5,197.5
7	41.25	5,238.75
8	20.625	5,259.375
9	10.3125	5,269.6875

1. Develop a formula for the *n*th term t_n of a sequence that represents the distance Betty walked in the *n*th segment of her trip.

2. Express the total distance S_n that Betty has traveled by the end of the *n*th segment of her trip in terms of the formula you developed for Problem 1. (Use symbols instead of specific numerical values.)

Shrinking Geometrically Key

1. Using the facts:

$$t_1 = \frac{1}{2}(5,280) = 2,640$$
$$t_{n+1} = \frac{1}{2}t_n \text{ for } 1 \le n \le 8$$

   Students should be able to come up with the formula:

$$t_n = \left(\frac{1}{2}\right)^n \times 5,280 \text{ for } 1 \le n \le 9$$

2. $S_n = t_1 + t_2 + \ldots + t_n$ for $1 \le n \le 9$.
   Students should recognize that S_n is a series. Some students may realize that

$$\begin{aligned} S_n &= \left(\tfrac{1}{2}\right)^1(5,280) + \left(\tfrac{1}{2}\right)^2(5,280) + \left(\tfrac{1}{2}\right)^3(5,280) + \ldots + \left(\tfrac{1}{2}\right)^n(5,280) \\ &= \left[\left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \ldots + \left(\tfrac{1}{2}\right)^n\right](5,280) \end{aligned}$$

   and wish for a more compact notation (which will preview sigma notation).

Sigma Notation

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Read the explanation of sigma notation below and then, on a separate sheet of paper, apply sigma notation to the series below. Use your notes and your Arithmetic Sequences and Series Graphic Organizer as a reference.

As our work on the Shrinking Geometrically worksheet made clear, some kind of shorthand for representing series is helpful. On that worksheet, we considered the following geometric series:

$$\left(\tfrac{1}{2}\right)^1(5{,}280) + \left(\tfrac{1}{2}\right)^2(5{,}280) + \left(\tfrac{1}{2}\right)^3(5{,}280) + \ldots + \left(\tfrac{1}{2}\right)^9(5{,}280)$$

It was tedious to write because we were adding together 9 terms, each of which had the same basic form:

$$\left(\tfrac{1}{2}\right)^k(5{,}280) \text{ for } 1 \le k \le 9$$

Mathematicians have devised a special notation—*sigma notation*—to deal with the problem. The word *sigma* comes from the capital Greek letter sigma (Σ), the equivalent of our capital *S* (for *sum*). Using sigma notation, we can write the following shorthand expression for the sum given above:

$$\sum_{k=1}^{9}\left[\left(\tfrac{1}{2}\right)^k(5{,}280)\right]$$

Here is how sigma notation works in general: Using the notation t_k to denote a number that depends on k, write:

$$\sum_{k=1}^{n} t_k = t_1 + t_2 + \ldots + t_{n-1} + t_n$$

Use sigma notation to represent each of the following series. Whenever possible, compute the value of the sum.

1. 1 + 2 + 3 + 4 + . . . + 100

2. (–18) + (–26) + (–34) + (–42) + (–50) + (–58) + (–66)

3. $10^1 + 10^2 + 10^3 + 10^4 + 10^5$

4. 12 + 5 + (–2) + (–9) + (–16) + (–23)

Sigma Notation Key

1. $\sum_{k=1}^{100} k = \frac{100}{2}(1+100) = 5{,}050$

2. $\sum_{k=1}^{7}[-18+(-8)(k-1)] = \frac{7}{2}[(-18)+(-66)] = \frac{7(-42)}{2} = -147$

3. $\sum_{k=1}^{5} 10^k = 111{,}110$

4. $\sum_{k=1}^{6}[12+(-7)(k-1)] = \frac{6}{2}[12+(-23)] = -33$

Geometric Sequences Vocabulary

1. Given the sequence $10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, \ldots$

   a. Each term is multiplied by a constant amount, ______________________, to produce the next term.

   b. Looking at it another way, if you take the ratio of any term to the one that came before it, you always obtain the same value, ______________, which is called the ______________________ ______________________, and denoted by *r*.

   c. Because each term is produced from the one that precedes it by a multiplicative factor, and because multiplication is often associated with geometric problems, the sequence is called a ______________________ ______________________.

2. Given the sequence $8 \times 10^1, 8 \times 10^2, 8 \times 10^3, 8 \times 10^4, 8 \times 10^5, 8 \times 10^6, 8 \times 10^7, \ldots$

   a. Each term is multiplied by a constant amount, ______________________, to produce the next term.

   b. Looking at it another way, if you take the ratio of any term to the one that came before it, you always obtain the same value, ______________, which is called the ______________________ ______________________, and denoted by *r*.

   c. Because each term is produced from the one that precedes it by a multiplicative factor, and because multiplication is often associated with geometric problems, the sequence is called a ______________________ ______________________.

3. Given the sequence 2,640; 1,320; 660; 330; 165; 82.5; 41.25; 20.625; 10.3125; . . .

   a. Each term is multiplied by a constant amount, ______________________, to produce the next term.

   b. Looking at it another way, if you take the ratio of any term to the one that came before it, you always obtain the same value, ______________, which is called the ______________________ ______________________, and denoted by *r*.

   c. Because each term is produced from the one that precedes it by a multiplicative factor, and because multiplication is often associated with geometric problems, the sequence is called a ______________________ ______________________.

Geometric Sequences Vocabulary Key

1. a. 10
   b. 10, *common ratio*
   c. *geometric sequence*

2. a. 10
   b. 10, *common ratio*
   c. *geometric sequence*

2. a. $\frac{1}{2}$
   b. $\frac{1}{2}$, *common ratio*
   c. *geometric sequence*

Contents

Half and Half Again

When we last discussed Betty's walk, we came up with the series

$$\sum_{k=1}^{9}\left[\left(\tfrac{1}{2}\right)^{k}(5,280)\right] = \left(\tfrac{1}{2}\right)^{1}(5,280) + \left(\tfrac{1}{2}\right)^{2}(5,280) + \ldots + \left(\tfrac{1}{2}\right)^{9}(5,280)$$

to describe how far Betty got after 9 segments of her Simon Says walk. Because the factor 5,280 appears in each term of the sum, it can be taken out as a common factor:

$$\sum_{k=1}^{9}\left[\left(\tfrac{1}{2}\right)^{k}(5,280)\right] = 5,280\sum_{k=1}^{9}\left(\tfrac{1}{2}\right)^{k} = (5,280)\left[\left(\tfrac{1}{2}\right)^{1} + \left(\tfrac{1}{2}\right)^{2} + \ldots + \left(\tfrac{1}{2}\right)^{9}\right]$$

In each segment of Betty's trip, she got closer and closer to home, but she never quite made it. To develop a general rule for the sum of a geometric series, let's explore the values of the expression

$$\sum_{k=1}^{n}\left(\tfrac{1}{2}\right)^{k} = \left[\left(\tfrac{1}{2}\right)^{1} + \left(\tfrac{1}{2}\right)^{2} + \ldots + \left(\tfrac{1}{2}\right)^{n-1}\right]$$

for several different values of n. Can you suggest a general rule for the sum?

n	$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k}$
1	
2	
3	
4	
5	
6	
7	
8	
9	

Half and Half Again Key

1. $\frac{1}{2}$
2. $\frac{3}{4}$
3. $\frac{7}{8}$
4. $\frac{15}{16}$
5. $\frac{31}{32}$
6. $\frac{63}{64}$
7. $\frac{127}{128}$
8. $\frac{255}{256}$
9. $\frac{511}{512}$

For the general rule, answers will vary but may include one of the following expressions:

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k} = \left(\frac{2^{n}-1}{2^{n}}\right) = 1-\left(\frac{1}{2}\right)^{n}$$

Summing Geometrically

Name: ______________________________ Period: ________________________ Date: ________________

Directions: On a separate sheet of paper, respond to each prompt as completely as you can. Show all your work and explain your answers.

In the Half and Half Again activity, we calculated the sum S_n of the series

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k$$

Now we want to develop another way to do it. Let's start by comparing the value of S_4 to the value of $\left(\frac{1}{2}\right)S_4$:

$$\begin{array}{rcccccccccc} S_4 & = & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & & \\ -\left(\frac{1}{2}\right)S_4 & = & & - & \frac{1}{4} & - & \frac{1}{8} & - & \frac{1}{16} & - & \frac{1}{32} \\ \hline [1-\left(\frac{1}{2}\right)]S_4 & = & \frac{1}{2} & + & 0 & + & 0 & - & 0 & - & \frac{1}{32} \end{array}$$

so that

$$\frac{1}{2}S_4 = \frac{1}{2} - \frac{1}{32}, \text{ so } S_4 = \frac{\left(\frac{1}{2} - \frac{1}{32}\right)}{\frac{1}{2}} = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}.$$

1. Demonstrate that, if you simplify the last expression by putting the terms over a common denominator, you get the value $\frac{2^4-1}{2^4}$ for S_4.
2. Perform the same calculations above to compare the value of S_6 to $\left(\frac{1}{2}\right)S_6$. Demonstrate that $S_6 = \frac{2^6-1}{2^6}$.
3. What happens when you perform the same calculations for S_9 ?

Summing Geometrically Key

1. $1 - \frac{1}{2^4} = \frac{2^4 - 1}{2^4}$

2.
$$\begin{array}{rcccccccccccccc}
S_6 & = & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} & + & \frac{1}{64} & & \\
-\left(\frac{1}{2}\right)S_6 & = & & - & \frac{1}{4} & - & \frac{1}{8} & - & \frac{1}{16} & - & \frac{1}{32} & - & \frac{1}{64} & - & \frac{1}{128} \\
\hline
\left[1-\frac{1}{2}\right]S_6 & = & \frac{1}{2} & + & 0 & + & 0 & + & 0 & + & 0 & - & 0 & - & \frac{1}{128}
\end{array}$$

Therefore,

$\frac{1}{2}S_6 = \frac{1}{2} - \frac{1}{128}$, so $S_6 = \frac{\left(\frac{1}{2} - \frac{1}{128}\right)}{\frac{1}{2}} = 1 - \frac{1}{64} = 1 - \frac{1}{2^6} = \frac{2^6 - 1}{2^6}$.

3. Students should complete the analogous table for S_9 and observe:

$S_9 = \frac{2^9 - 1}{2^9}$.

Contents

General Geometric Series

Suppose that $\sum_{k=1}^{n} ar^{k-1}$ is a geometric series of n terms. Let's try to find a general formula for the sum. As before, let's use the shorthand:

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \ldots + ar^{n-1}$$

Step 1

As we did yesterday, let's look at the relationship between rS_n and S_n:

$S_n =$	a	$+ ar$	$+ ar^2$	$+ \ldots$	$+ ar^{n-1}$		
$-rS_n =$		$- ar$	$- ar^2$	$- \ldots$	$- ar^{n-1}$	$- ar^n$	
$(1-r)S_n =$	a	$+ 0$	$+ 0$	$+ \ldots$	$+ 0$	$- ar^n$	

Step 2

Use the table to find a formula for S_n. What happens when $r = 1$?

General Geometric Series Key

Step 2

$(1 - r)S_n = a - ar^n = a(1 - r^n)$

When $r \neq 1$, $S_n = a\dfrac{(1 - r^n)}{1 - r}$.

When $r = 1$, $S_n = \sum_{k=1}^{n} a1^k = \sum_{k=1}^{n} a = na$.

Geometric Sequences and Series Graphic Organizer

Name: ______________________________ Period: ________________________ Date: ________________

Geometric Sequence

Definition

Examples

Nonexamples

Rules & Formulas

Geometric sequences are expressed in the form
$t_1, t_2, t_3, \ldots t_n$.

1. The ratio between each term and the one preceding it is called the ____________________________.
2. It is found using the formula

   $r =$

3. A general expression for the kth term of a geometric sequence is:

   $t_k =$

Practice

Use the kth term formula to find the general rule for generating a term in the sequence
–3, 12, –48, 192, . . .

Geometric Series

Definition

Examples

Nonexamples

Rules & Formulas

Instead of adding each term in an geometric series, we can find the sum by using the following formula:

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \_\_\_\_\_$$

$a =$ ____________________

The formula is valid for the following values of r:

______________________.

Practice

Find the sum of the first 8 terms of the series
6 + 12 + 24 + 48 + . . .
(*Hint*: Find the values of each variable listed above and calculate S_n.)

Geometric Sequences and Series Graphic Organizer Key

Geometric Sequence

Definition

An ordered list of numbers in which the ratio between each term and the one preceding it is constant.

Examples

Answers will vary, but may include:

- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
- $10, 10^2, 10^3, \ldots$
- $1, 1, 1, 1, \ldots$

Nonexamples

Answers will vary, but may include:

- $1, 2, 3, 4, \ldots$
- sequence of primes
- Michael's Simon Says sequence

Rules & Formulas

Geometric sequences are expressed in the form $t_1, t_2, t_3, \ldots t_n$.

1. The ratio between each term and the one preceding it is called the <u>common ratio</u>.
2. It is found using the formula
   $$r = \frac{t_{k+1}}{t_k}$$
3. A general expression for the kth term of a geometric sequence is $t_k = t_1 r^{k-1}$.

Practice

Use the kth term formula to find the general rule for generating a term in the sequence
$-3, 12, -48, 192, \ldots$

$$t_k = (-3)(-4)^{k-1}$$

Geometric Series

Definition

The sum of the terms of an geometric sequence.

Examples

Answers will vary, but may include:

- $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$
- $10 + 10^2 + 10^3$
- $1 + 1 + 1 + 1$

Nonexamples

Answers will vary, but may include:

- $1 + 2 + 3 + 4$
- series of primes
- Michael's Simon Says series

Rules & Formulas

Instead of adding each term in an geometric series, we can find the sum by using the following

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right)$$

formula:
$a = t_1$

The formula is valid for the following values of r:

Any real $r \neq 1$.

Practice

Find the sum of the first 8 terms of the series
$6 + 12 + 24 + 48 + \ldots$
(*Hint*: Find the values of each variable listed above and calculate S_n.)

$$\sum_{k=1}^{8} 6r^{k-1} = 6\left(\frac{1-2^8}{1-2}\right) \quad a = 6,\ r = 2$$
$$= 6(255) = 1{,}530$$

Geometric Sequences and Series Practice

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Answer each of the following questions on a separate sheet of paper. Use your graphic organizers as references, but be sure to indicate what facts and formulas you use and to explain your answers.

1. Given the sequence 3, –6, 12, –24, . . .
   a. Why is it a geometric sequence? What is the common ratio r ?
   b. Write a formula for the kth term of the sequence.
   c. What is the sum of the first 7 terms of the sequence?
2. Given the sequence 1, –1, 1, –1, . . .
   a. Is it an arithmetic sequence or a geometric sequence? Why?
   b. Write a formula for the kth term of the sequence.
   c. If n is odd, what is the sum of the first n terms of the sequence? What happens if n is even?
3. Given the series 2 + 4 + 8 + 16 + . . .
   a. Why is it a geometric series?
   b. Find the sum S_{10}.
   c. Suppose that S_n = 4,094. What is the value of n ?
4. Calculate the sum of the series $\sum_{k=1}^{n}\left(-\frac{1}{2}\right)^k$ when n = 3 and 4. Express your answer as fractions; do not use a calculator. (Hint: Write the kth term of the series in the form ar^{k-1}.)

Geometric Sequences and Series Practice Key

1. a. It is geometric because the ratio (common ratio *r*) of each term to its predecessor is equal to –2.

   b. $a_{k} = 3(-2)^{k-1}$

   c. $$\sum_{k=1}^{7} 3(-2)^{k-1} = 3\left(\frac{1-(-2)^7}{1-(-2)}\right) = 3\left(\frac{129}{3}\right) = 129$$

2. a. It is not arithmetic because the difference between a term and its predecessor is not constant. (The differences alternate between –2 and +2.) It is geometric because the ratio of each term to its predecessor is constant and equal to –1.

   b. $a_{k} = (-1)^{k-1}$

   c. $$\sum_{k=1}^{n} (-1)^{k-1} = 1\frac{\left(1-(-1)^n\right)}{1-(-1)} = \frac{1}{2}\left[1-(-1)^n\right]$$

   When *n* is odd, the sum is 1; when *n* is even, the sum is 0.

3. a. It is geometric because the ratio of each term to its predecessor is 2.

   b. $$S_{10} = \sum_{k=1}^{10} 2\left(2^{k-1}\right) = 2\left(\frac{1-2^{10}}{1-2}\right) = 2^{11} - 2 = 2{,}046$$

   c. $$4{,}094 = S_n = 2\left(\frac{1-2^n}{1-2}\right) = 2^{n+1} - 2$$

   $$4{,}096 = 2^{12} = 2^{n+1}$$

   $$n = 11$$

$$S_n = -\frac{1}{2}\left[\frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}\right] = -\frac{1}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

4. $a = -\frac{1}{2}$, $r = -\frac{1}{2}$

   $S_3 = -\frac{3}{8}$, $S_4 = -\frac{5}{16}$

Contents

Sequences, Series, and Salaries

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Respond to the following prompts.

For a 30-day temporary job, you were offered 2 salary options.

- Plan A: $1,000 a day.
- Plan B: $.01 the 1st day, $.02 the 2nd day, $.04 the 3rd day, and so on through the 30th day. (This plan pays double what you received the day before.)

1. Without calculating the pay for each plan, make a conjecture about which plan will pay the most.

2. Complete the table at right.

3. With each plan, how much money will you have earned at the end of 30 days? (It may be useful to use a summation formula.)

Day	Plan A Earnings	Plan B Earnings
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		
27		
28		
29		
30		

4. Which pattern is arithmetic? Which is geometric? Justify your answers.

5. On which day were the cumulative totals approximately equal? (Hint: Use a summation formula.)

6. Why might someone choose Plan A over Plan B?

7. Discuss the advantages and disadvantages of each plan.

Sequences, Series, and Salaries Key

1. Answers will vary.
2. See table at right.
3. Plan A will have earned $30,000. Plan B will have earned

$$\sum_{n=1}^{30} .01(2^{n-1}) = .01\left(\frac{1-2^{30}}{1-2}\right) = \$10,737,418.23$$

4. Plan A is arithmetic with a difference of 0 or geometric with a ratio of 1. Plan B is geometric with a ratio of 2.
5. Since each day in Plan A is $1,000, it is necessary to look at the days in Plan B where earnings are greater than $1,000. On day 21, Plan A would be $21,000 and Plan B would be $.01\left(\frac{1-2^{21}}{1-2}\right) = \$20,971.51$.
6. Answers will vary, but should acknowledge that Plan A gives more money up front than Plan B.
7. Plan A: advantages = more money up front; disadvantages = less money total.
   Plan B: advantages = much greater total earnings; disadvantages = takes more time to earn the same amount as Plan A.

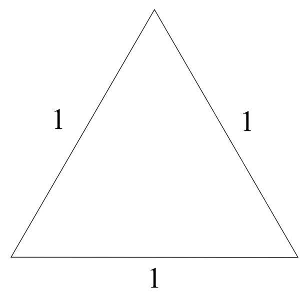
Day	Plan A Earnings	Plan B Earnings
1	$1,000	$.01
2	$1,000	$.02
3	$1,000	$.04
4	$1,000	$.08
5	$1,000	$.16
6	$1,000	$.32
7	$1,000	$.64
8	$1,000	$1.28
9	$1,000	$2.56
10	$1,000	$5.12
11	$1,000	$10.24
12	$1,000	$20.48
13	$1,000	$40.96
14	$1,000	$81.92
15	$1,000	$163.84
16	$1,000	$327.68
17	$1,000	$655.36
18	$1,000	$1,310.72
19	$1,000	$2,621.44
20	$1,000	$5,242.88
21	$1,000	$10,485.76
22	$1,000	$20,971.52
23	$1,000	$41,943.04
24	$1,000	$83,886.08
25	$1,000	$167,772.16
26	$1,000	$335,544.32
27	$1,000	$671,088.64
28	$1,000	$1,342,177.28
29	$1,000	$2,684,354.56
30	$1,000	$5,368,709.12

Sequences, Series, and Patterns

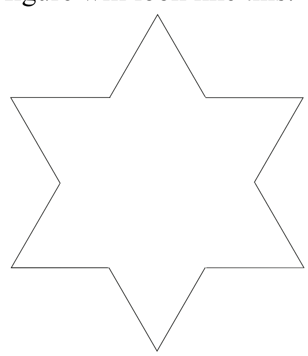
Name: ______________________________ Period: ________________________ Date: ________________

Directions: Read the directions explaining how to create a Koch snowflake. Then, respond to the prompts that follow.

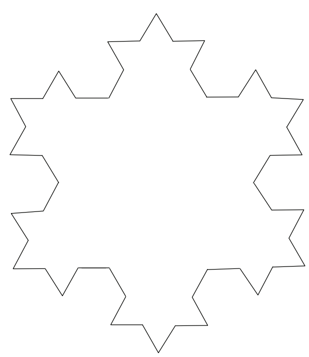
Here is an equilateral triangle with sides of length 1 unit.



Remove the middle $\frac{1}{3}$ of each side and replace each section with 2 sides of an equilateral triangle whose sides are the same length as the removed segment. The figure will look like this:



Repeating the process, take out the middle $\frac{1}{3}$ of each segment and replace each section with two sides of an equilateral triangle to create this figure:



1. Given that the length of each side of the original equilateral triangle is 1 unit, what is the perimeter of the 1st shape above? What is the perimeter of the 2nd? What is the perimeter of the 3rd? Can you see a pattern? Why does this pattern occur?

2. Without drawing, predict the perimeter of the next shape if it is created using the same process as before.

3. What would be the perimeter of the 7th figure in the pattern?

4. Predict the perimeter of any shape in the series. Justify your answer. Write a formula to find the perimeter of any figure in this pattern.

Sequences, Series, and Patterns Key

1. The perimeter of the 1st shape is 3 units. The perimeter of the 2nd shape is 4 units. The perimeter of the 3rd shape is $5\frac{1}{3}$ (or $\frac{16}{3}$ units). The pattern is to multiply the perimeter of the previous shape by $\frac{4}{3}$ each time.
2. The perimeter of the next shape is $\frac{16}{3} \cdot \frac{4}{3}$, or $\frac{64}{9}$ units ($7\frac{1}{9}$ units).
3. $3\left(\frac{4}{3}\right)^6 = \frac{4096}{243} = 16\frac{208}{243} \approx 16.86$ units.
4. $a_n = a_1\left(r^{n-1}\right) = 3\left(\frac{4}{3}\right)^{n-1}$

Reflecting on Patterns

Name: ____________________________ Period: ______________________ Date: _______________

Directions: Fill out the worksheet to help you summarize and reflect on what you have learned.

Essential Questions

1. Why are patterns useful in our everyday lives?

2. How do you know when an observed pattern forms a sequence?

3. How are patterns useful in making predictions?

Important Formulas

- Sequences/*n*th Term

- Summation Formulas

Important Things to Remember

Questions I Still Have

Sequences and Series Review

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Is the sequence arithmetic, geometric, or neither? Explain your answer.

1. –1, 1, 3, 5, . . .

2. 3, 8, 9, 12, . . .

3. 2, 4, 8, 16, . . .

4. –6, –1, 4, 9, . . .

Directions: Find the *n*th term of the following sequences.

Arithmetic

5. From the sequence: 3, 7, 11, 15, . . .

6. From the values in the formula: $a_{12} = 18$ and $d = 2$

7. From 2 terms in the sequence: $a_4 = 18$ and $a_{10} = 48$

Geometric

8. From the sequence: 3, 9, 27, 81, . . .

9. From the values in the formula: $a_1 = 64$ and $r = 0.25$

10. From 2 terms in the sequence: $a_3 = 18$ and $a_6 = –486$

Arithmetic

11. Find the sum of the first 20 terms of the series 40 + 37 + 34 + 31 + . . .

12. Given the series 40 + 37 + 34 + 31 + . . . , find the value of n if $S_n = 195$.

Geometric

13. Find the sum of the first 8 terms of the series 1 + (–4) + 16 + (–64) + . . .

14. Given the series 1 + (–4) + 16 + (–64) + . . . , find the value of n if $S_n = –819$.

Directions: Express each arithmetic or geometric series in sigma notation. What is the sum of each finite series?

15. 3 + 8 + 13 + 18

16. 2 + 8 + 32 + 128

Sequences and Series Review Key

1. Arithmetic, add 2 each time.
2. Neither, there is no common difference or common ratio.
3. Geometric, multiply by 2 each time.
4. Arithmetic, add 5 each time.
5. $a_n = 3 + (n - 1)4 = 4n - 1$
6. $a_{12} = 18$ and $d = 2$, $a_n = -4 + 2(n - 1) = 2n - 6$
7. $a_4 = 18$ and $a_{10} = 48$, $a_n = 3 + (n - 1)5 = 5n - 2$
8. $a_n = 3(3)^{n-1} = 3^n$
9. $a_1 = 64$ and $r = 0.25$; $a_n = 64(0.25)^{n-1}$
10. $a_3 = 18$ and $a_6 = -486$; $a_n = 2(-3)^{n-1}$
11. $S_n = \frac{20}{2}[40 + 40 + (-3)(20 - 1)] = 230$
12. $195 = \frac{6}{2}[40 + 40 + (-3)(n - 1)]$

    $n = 6$
13. $S_n = 1\left[\frac{1-(-4)^8}{1-(-4)}\right] = -13{,}107$
14. $-819 = 1\left[\frac{1-(-4)^8}{1-(-4)}\right] = -4{,}095 = 1 - (-4)^8$

    $n = 6$
15. $\sum_{n=1}^{4} 3 + (n-1)5 = 5n - 2 = 42$
16. $\sum_{n=1}^{4} 2(4)^{n-1} = 170$

Sequences and Series Test

Name: ____________________________ Period: ______________________ Date: _______________

Directions: Is the sequence arithmetic, geometric, or neither? Explain your answer.

1. 2, –4, –10, –16, . . .

2. 0, 4, 9, 15, 22, . . .

3. 6, 4, $\frac{8}{3}$, $\frac{16}{9}$, . . .

4. –2, –0.5, 1, 2.5, . . .

For Problems 5–6 write a formula for the *n*th term of the arithmetic sequence. Then, find a_{25}.

5. $d = -6$, $a_{12} = -4$

6. $a_7 = 34$, $a_{18} = 122$

7. Given the arithmetic series 34 + 31 + 28 + 25 + 22 + . . .
   a. Find the sum of the first 32 terms.

   b. Find n for $S_n = -12$.

For Problems 8–9, write the rule for the *n*th term of the geometric sequence, then find a_9.

8. 1, –6, 36, –216, . . .

9. $a_2 = -30$, $a_5 = 3{,}750$

10. Given the geometric series 7 + (–21) + 63 + (–189) + . . .

    a. Find the sum of the first 18 terms.

    b. Find *n* for $S_n = 3{,}829$.

For Problems 11–12, express the finite series in sigma notation, then evaluate the expressions.

11. 48 + 24 + 12 + 6

12. 3 + 3.5 + 4 + 4.5

13. A local pilot has offered to drop the game ball for the last football game from his airplane. During the 1st second, the football falls 4.9 m. During the next second, it falls 14.7 m; during the 3rd, 24.5 m; and during the 4th, 34.3 m. If this pattern continues, how far will the football fall during the 15th second? What is the total distance the football has fallen after 15 seconds?

Sequences and Series Test Key

1. Arithmetic, $d = –6$
2. Neither: there is no common difference or common ratio.
3. Geometric, $r = \frac{2}{3}$
4. Arithmetic, $d = 1.5$
5. $a_n = 62 + (n – 1)(–6) = –6n + 68$

   $a_{25} = 62 + 24(–6) = –82$
6. $a_n = –14 + (n – 1)(8) = 8n – 22$

   $a_{25} = –14 + (24)(8) = 178$
7. a. $S_{32} = \frac{32}{2}[34 + (34 - 3 \cdot 31)]$

   b. $\frac{n}{2}[34 + 34 - 3(n - 1)$

   $n = 24$
8. $a_n = 1(–6)^{n-1}$

   $a_9 = 1,679,616$
9. $a_n = 6(–5)^{n-1}$

   $a_9 = 2,343,750$
10. a. $S_{18} = 7\left(\frac{1-(-3)^{18}}{1-(-3)}\right) = -677,985,854$

    b. $3,829 = 7\left(\frac{1-(-3)^{n}}{1-(-3)}\right)$

    $n = 7$
11. $\sum_{n=1}^{4} 48\left(\frac{1}{2}\right)^{n-1} = 90$
12. $\sum_{n=1}^{4} (0.5n + 2.5) = 15$
13. $a_{15} = 4.9 + (15 – 1)(9.8) = 142.1$ m

    $S_{15} = \left(\frac{(4.9 + 142.1)15}{2}\right) = 1,102.5$ m

Contents

Famous Fibonacci

Name: ____________________________ Period: ______________________ Date: _______________

Directions: Respond to the prompts below.

The Fibonacci Sequence

Leonardo Fibonacci was an Italian mathematician who lived about 1200 A.D. He is best known for his work with a sequence of numbers now known as the Fibonacci numbers:

0, 1, 1, 2, 3, 5, 8, 13, . . .

The *n*th term in the sequence is often denoted F_n, in honor of Fibonacci, and the sequence normally begins with $n = 0$, so that $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and so forth.

1. Is the Fibonacci sequence arithmetic, geometric, or neither? Justify your answer.
2. Can you identify a pattern in the Fibonacci sequence?
3. When the *n*th term of a sequence is determined by one or more of the preceding terms, the sequence is called recursive. Explain why the Fibonacci sequence is recursive, and give a formula for the *n*th term. For which values of *n* is the formula valid?

Connections

4. What is the connection between the Golden Ratio, the Golden Rectangle, and the Fibonacci sequence? (This one may require some research!)

Famous Fibonacci Key

1. Neither: it has neither common difference nor common ratio.
2. For $n \geq 2$, the nth term in the sequence is equal to the sum of the two preceding terms.
3. It is recursive because, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.
4. Two positive numbers a and b are said to be in the golden ratio if $b > a$ and $\frac{a+b}{b} = \frac{b}{a}$. Using the quadratic formula, it can be shown that the 2 ratios must always equal the constant value $\frac{1+\sqrt{5}}{2} \approx 1.618$. The value is an irrational number denoted by the Greek letter φ (phi) and referred to as the Golden Ratio. A rectangle in which the ratio of the sides is equal to φ is called a Golden Rectangle, which artists and architects throughout recorded history have considered aesthetically pleasing. It can be shown that $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$. It can also be shown that as n increases, the ratio $\frac{F_n}{F_{n-1}}$ approaches φ.

Real-World Patterns

Directions: What kind of sequence (arithmetic, geometric, or neither) is suggested by each problem?

1. To improve his health and fitness, Walter plans to walk every day. During the first week, he will walk around the perimeter of the square block in which he lives. Each subsequent week, he intends to walk one block further west of the original loop. (See Figure 1.) He will keep track of the distance he walks each day on a calendar, expressing the distance in units, where one unit equals one side of a square block.

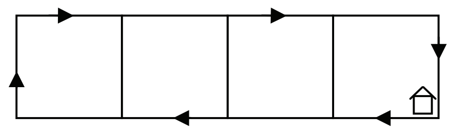


Figure 1

2. In a different plan, Walter will again begin the first week by walking clockwise around the square block on which he lives. Each subsequent week, Walter will walk the length of one additional block in each direction before he returns home. (See Figure 2.) Each week his route encloses consecutively larger square paths. On his calendar, Walter will record the number of square blocks within the perimeter of his path.

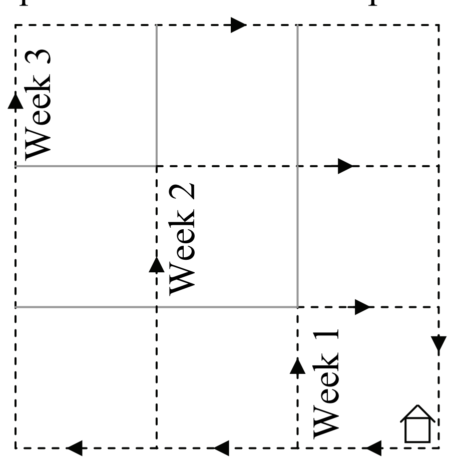


Figure 2

3. In Walter's final plan, he again begins the first week by walking clockwise around the square block on which he lives. As usual, he keeps track of the distance he walks each day on a calendar, expressing the distance in units, where one unit equals one side of a square block. Each subsequent week, Walter doubles the number of units walked the week before.

Real-World Patterns Key

1. Arithmetic
2. Arithmetic
3. Geometric

Real-Life Applications of Sequences and Series

Name: ______________________________ Period: ________________________ Date: ________________

Directions: Solve the following real-world problems.

1. Suppose you buy a \$3,000 high-definition television on layaway by making a down payment of \$300 the first month and then paying \$75 each month thereafter. Write a formula to represent the total amount of money that has been paid on the television set by the end of the *n*th month. How much more will you have to pay after 9 months?

2. In order to help reduce United States dependency on foreign oil, Alaska is investigating the possibility of drilling a well in the southern part of the state. The costs to drill are \$1,500 for the 1st foot, \$1,750 for the 2nd foot, \$2,000 for the 3rd foot, and so on. How much will it cost to drill a 1,250 ft well?

3. You are stacking soup cans for a display in a grocery store. Your manager wants you to stack 136 cans in layers, with each layer after the first having one fewer can than the layer below it. The top layer should contain one can. In order to stack all 136 soup cans, how many cans must be in the 1st layer?

4. Your parents are buying you a car for graduation. The original cost of the automobile is \$18,000, and its yearly depreciation rate is 15% of its value at the beginning of the year. Write a rule for the value (a_n) of the automobile at the beginning of the *n*th year. After how many years will the automobile be worth less than \$10,000 ?

Real-Life Applications of Sequences and Series Key

1. $300 + 75(n - 1)$; $3{,}000 - [300 + 75(9 - 1)] = \$2{,}100$
2. $S_{1250} = \frac{1{,}250}{2}[1{,}500 + 1{,}500 + 250(1{,}250 - 1)] = \$197{,}031{,}250$
3. Assuming the first term is 1 and the common difference is 1: $136 = n\frac{(1+n)}{2}$, $n = 16$
4. $10{,}000 = 18{,}000(.85)^{n-1}$; 4 years

Secondary ACT Course Standards

A *primary course standard*

- is the central focus of the unit and
- is explicitly assessed in an embedded assessment and/or in the summative assessment.

A *secondary course standard*

- is less important to the focus of the unit, but is one that students need to know and use when completing activities for this unit and
- may or may not be explicitly assessed by the summative assessment or an embedded assessment.

ACT Course Standards considered primary for this unit are listed on pp. 1–2. Below is a list of secondary ACT Course Standards associated with this unit.

Selected Secondary ACT Course Standards

A. 1. Prerequisites

a. Identify properties of real numbers and use them and the correct order of operations to simplify expressions

b. Multiply monomials and binomials

d. Solve single-step and multistep equations and inequalities in one variable

f. Write linear equations in standard form and slope-intercept form when given two points, a point and the slope, or the graph of the equation

g. Graph a linear equation using a table of values, *x*- and *y*- intercepts, or slope-intercept form

j. Use inductive reasoning to make conjectures and deductive reasoning to arrive at valid conclusions

ACT Course Standards Measured by Assessments

This table presents at a glance how the ACT Course Standards are employed throughout the entire unit. It identifies those standards that are explicitly measured by the embedded and unit assessments. The first column lists ACT Course Standards by a three-character code (e.g., A.1.a.); columns 2–10 on this and 2–9 on the next page list the assessments.

Coded ACT Course Standard	Embedded Assessments								
	Prime Factors	Number Patterns	Algebra Skills	Station Problems	Group Participation and Collaboration Rubric	Simon Says	Simon Says 2	Movie Time	Even and Odd
A.1.a.	X	X	X	X					
A.1.b.			X						
A.1.d.			X						
A.1.f.			X						
A.1.g.			X						
A.1.j.				X		X	X	X	X
B.1.d.					X	X	X	X	X
B.1.e.	X	X		X	X				
B.1.f.					X	X		X	
B.1.g.									
H.2.a.					X		X	X	X
H.2.b.					X				
H.2.c.					X		X	X	X
H.2.d.					X	X	X	X	
H.2.e.					X				

Coded ACT Course Standard	Embedded Assessments							Unit Assessment
	Arithmetic Sequences and Series Practice	Growing Geometrically	Shrinking Geometrically	Geometric Sequences and Series Practice	Sequences, Series, and Salaries	Sequences, Series, and Patterns	Sequences and Series Review	Sequences and Series Test
A.1.a.								
A.1.b.								
A.1.d.								
A.1.f.								
A.1.g.								
A.1.j.	X	X	X	X	X	X	X	X
B.1.d.	X	X		X	X	X	X	X
B.1.e.								
B.1.f.		X			X	X		
B.1.g.								X
H.2.a.	X	X		X	X	X	X	X
H.2.b.	X			X	X	X	X	X
H.2.c.	X	X		X	X		X	X
H.2.d.		X			X	X		X
H.2.e.			X	X			X	X